1. Power series.

1.1. Definition: a series of functions of the form $\sum_{n=1}^{\infty} a_n x^n$ where $a_1, a_2, \ldots$, are given (complex) numbers, is called a power series (in $x$).

Partial sums: $S_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial of degree (at most) $n$.

Example. For $\sum_{n=1}^{\infty} n x^n$, the coefficient $a_n$ is given by $a_n = \begin{cases} 0, & n \text{ is not a square} \\ \sqrt{n}, & n \text{ is a square} \end{cases}$. In this example $S_5(x) = x + 2x^4$.

1.2. Definition: the set of real (or complex) values of $x$ for which the power series converges is called the domain of convergence. We’ll denote it by $D$: $D = \{x \in \mathbb{R} : \sum_n a_n x^n \text{ convergent}\}$.

Note: $0 \in D$, so $D$ is never empty.

2. Radius of convergence

2.1. Definition: the radius of convergence $R$ is the positive number (or $+\infty$) given by

$$\frac{1}{R} = \limsup_n |a_n|^{1/n}$$

2.2. Theorem [Cauchy-Hadamard]. The power series $\sum_{n=1}^{\infty} a_n x^n$:

a) converges absolutely for $|x| < R$.

b) diverges for $|x| > R$

c) Moreover, the convergence is uniform on $|x| \leq R - \delta$, for any $\delta > 0$.

Note: the case of $|x| = R$ is not covered by the theorem, and needs to be analyzed separately.

Note: if $R = +\infty$ then $D = \mathbb{R}$.

3. How to compute $R$.

In most of the specific cases $|a_n|^{1/n}$ is actually convergent, in which case $\limsup = \lim$.

3.1. First method: keep in mind the following facts:

- (Average) $x_n \to L \Rightarrow \frac{1}{n} (x_1 + x_2 + \cdots + x_n) \to L$
- (Multiplicative analogue) $|x_n| > 0$, $x_n \to L \Rightarrow \lim_n (x_1 x_2 \cdots x_n)^{1/n} = L$.
- Corollary: if $x_n > 0$ and $x_{n+1}/x_n \to L$, then $x_n^{1/n} \to L$.

3.2. Second method: let $b_n = \log(|a_n|^{1/n}) = \frac{1}{n} \log |a_n|$. Check if $b_n$ has a limit, then $|a_n|^{1/n} \to e^{\lim b_n}$.

3.3. Example. $\sum_{n=1}^{\infty} (-1)^n n 2^n x^n$.

Coefficients: $a_n = (-1)^n n 2^n$; $|a_n|^{1/n} = 2 n^{1/n}$. What is $\limsup_{n \to +\infty} 2 n^{1/n}$?

Take log: $b_n = \log |a_n|^{1/n} = \ln 2 + \frac{\ln n}{n} \to \ln 2$. Then $|a_n|^{1/n} \to e^{\ln 2} = 2$.

Radius: $1/R = \lim |a_n|^{1/n} = 2 \Rightarrow R = 1/2$.

Hence $(-1/2, 1/2) \subseteq D$.

Check "endpoints" separately:

At $x = -1/2$: $\sum a_n x^n = \sum n$ divergent.

At $x = 1/2$: $\sum a_n x^n = \sum (-1)^n n$ divergent.

Therefore $D = (-1/2, 1/2)$.

3.4. Example. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$.

Coefficients: $a_n = 1/n!$, $|a_n|^{1/n} = 1/(n!)^{1/n}$.

Take log: $b_n = \log |a_n|^{1/n} = -\frac{\log n}{n}$. Not so clear what $\lim b_n$ is.

Alternative: $|a_{n+1}|/|a_n| = \frac{1/(n+1)!}{1/n!} = \frac{1}{n+1} \to 0$, hence $|a_n|^{1/n} \to 0$ as well.

Conclusion: $R = +\infty$, $D = \mathbb{R}$.

3.5. Example. $\sum n x^n$. Here $|a_n|^{1/n} = \begin{cases} n^{1/2n}, & n \text{ is a square} \\ 0, & \text{otherwise} \end{cases}$. In this case $|a_n|^{1/n}$ is not convergent, but $\limsup |a_n|^{1/n} = 1$. (Justify this step.) So $R = 1$. It is not hard to see that $D = (-1, 1)$.
4. HOW TO USE POWER SERIES

4.1. Assume \( R \) is the radius of convergence of the power series \( \sum a_n x^n \). Since \( n^{1/n} \to 1 \), it is not hard to see that the two series \( \sum_{n=1}^{\infty} na_n x^{n-1} \) and \( \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \) have the same radius of convergence \( R \). Since the power series converges uniformly on compact intervals of the form \([-r, r] \subset (-R, R)\), we have

**Theorem.** Let \( f(x) = \sum a_n x^n \) for \( x \in D \). Then:
a) \( f(x) \) is differentiable on \((-R, R)\) and \( f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} \), \( \forall x \in (-R, R) \).
b) \( \int_0^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \), \( \forall x \in (-R, R) \).

**Note:** applying this theorem repeatedly we see that the sum of a power series is a smooth (infinitely many times differentiable) function inside the radius of convergence, \( f \in C_\infty ((-R, R)) \).

4.2. Example. Let \( f(x) = \sum x^n \), for \( x \in \mathbb{R} \). Then \( f \) is \( C^1 \) on \( \mathbb{R} \), and \( f'(x) = \sum_{n=1}^{\infty} \frac{n}{n!} x^{n-1} \), that is \( f'(x) = f(x) \). Moreover \( f(0) = 1 \). The only function that satisfies this equation is \( f(x) = e^x \). In particular, \( \sum_{n=0}^{\infty} \frac{1}{n!} = e \).

4.3. Example. Let \( f(x) = \sum (-1)^n x^{2n} \), for \(|x| < 1 \) (\( R = 1 \)). Then clearly \( f(x) = \frac{1}{1+2x} \).

Integrate: \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \int_0^x \frac{1}{1+2t} dt = \tan^{-1}(x) \), for \(|x| < 1 \).

4.4. Example. \( \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \), for \(|x| < 1 \).

5. Abel’s theorem

5.1. Note that if one could take \( x = 1 \) in the previous two examples, we would obtain nice identities:

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \]

This step is justified by the following theorem:

**Theorem**[Abel]. Let \( R \) the radius of convergence of \( \sum_{n=0}^{\infty} a_n x^n \), and let \( f(x) \) the sum of the power series inside the radius of convergence. If \( R \in D \), then \( \lim_{x \to R^-} f(x) = \sum_{n=0}^{\infty} a_n R^n \). (The limit \( f(R^-) \) exists and equals the sum of the series at \( x = R \).)

Note: the theorem is works just the same for \(-R \) instead of \( R \).