LANDAU NOTATION AND DERIVATIVE

1. Landau Notation

1.1. Assume $P(x)$ a statement that depends on the variable $x$.
"As $x \to x_0$, eventually $P(x)$" means that there exists an open neighborhood $V$ of $x_0$ such that $P(x)$ is true for every $x \in V$.
"As $n \to \infty$, eventually $P(n)$" means that $\exists N > 0$ such that for $n > N$, $P(n)$ is true.

Example: $\lim_{x \to x_0} f(x) = L$ is the statement that $\forall \epsilon > 0$, eventually $|f(x) - L| < \epsilon$. In this case "eventually" means $\exists \delta > 0$ (possibly depending on $\epsilon$), such that for $|x - x_0| < \delta$.

Remark: assume $g(x)$ is smaller (in absolute value) than any given fraction of $f(x)$, up to a constant multiple.

Note: with this definition, if $g(x)$ might actually vanish at $x_0$ (or in a neighborhood of $x_0$), in which case the quotient $f(x_0)/g(x_0)$ is not well-defined.

Remark: this is almost the same as saying that the quotient $f(x)/g(x)$ is eventually bounded as $x \to x_0$, the only problem with this definition being that $g(x)$ might actually vanish at $x_0$ (or in a neighborhood of $x_0$), in which case the quotient $f(x_0)/g(x_0)$ is not well-defined.

1.2. Assume $f(x)$ and $g(x)$ are two functions defined in a neighborhood of $x_0$. Moreover, let’s assume that $g(x)$ is reasonably simple, while $f(x)$ is a function whose behavior (near $x_0$) we want to understand in terms of that of $g(x)$.

1.3. Definition. "As $x \to x_0$, $f(x) = O(g(x))$" means that there exists $C > 0$ such that eventually $|f(x)| \leq C |g(x)|$, as $x \to x_0$.

Equivalent meaning: there exists an open neighborhood of $x_0$ where $f$ (in absolute value) is no bigger than $g$, up to a constant multiple.

Equivalent definition: $\exists C > 0$ and $\delta > 0$ such that $|f(x)| \leq C |g(x)|$ for $|x - x_0| < \delta$.

Note: this is almost the same as saying that the quotient $f(x)/g(x)$ is eventually bounded as $x \to x_0$, the only problem with this definition being that $g(x)$ might actually vanish at $x_0$ (or in a neighborhood of $x_0$), in which case the quotient $f(x_0)/g(x_0)$ is not well-defined.

Remark: with this definition, if $g(x_0) = 0$ and $f(x) = O(g(x))$ as $x \to x_0$, then $f(x_0) = 0$.

1.4. Definition. "As $x \to x_0$, $f(x) = o(g(x))$" means $\forall \epsilon > 0$, eventually $|f(x)| \leq \epsilon |g(x)|$ as $x \to x_0$.

Equivalent meaning: $f$ is smaller (in absolute value) than any given fraction of $g$ in a small enough neighborhood of $x_0$.

Equivalent definition: $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that $|f(x)| \leq \epsilon |g(x)|$ for $|x - x_0| < \delta(\epsilon)$.

Note: this is almost the same as saying that $f(x)/g(x) \to 0$ as $x \to x_0$, the problem with this formulation being once more that $g(x)$ might actually vanish at $x_0$ (or in a neighborhood of $x_0$).

Remark: assume $g(x_0) = 0$ but $g \neq 0$ on $V - \{x_0\}$, where $V$ is some neighborhood of $x_0$. Then $f(x) = o(g(x))$ if and only if $f(x_0) = 0$ and $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$.

1.5. Definition. "As $x \to +\infty$, $f(x) = O(g(x))$" means that there exists $C > 0$ and $N > 0$ such that $|f(x)| \leq C |g(x)|$ for all $x > N$.

1.6. Definition. "As $x \to +\infty$, $f(x) = o(g(x))$" means $\forall \epsilon > 0$, $\exists N(\epsilon) > 0$ such that $|f(x)| \leq \epsilon |g(x)|$ for $x > N(\epsilon)$.

1.7. Examples.
- $f$ is continuous at $x_0$ if and only if $f(x) = f(x_0) + o(1)$ as $x \to x_0$.
- If $f(x) = o(g(x))$ as $x \to x_0$, then $f(x) = O(g(x))$ as $x \to x_0$
- $x - x_0 = o(1)$ as $x \to x_0$
- $(x - x_0)^2 = o(x - x_0)$
- If $k < N$, $x^k = o(x^N)$ as $x \to +\infty$
- If $k < N$, $x^N = o(x^k)$ as $x \to 0$
- For any $a > 0$, $\ln x = o(x^a)$ as $x \to +\infty$
- $x = O(\sqrt{x})$ as $x \to 0$
- $(x - x_0)^2 = o(x - x_0)$ as $x \to x_0$
- $x - 1 = O((x - 1)^2)$ as $x \to 0$
- $\sin(x) = O(x)$ as $x \to 0$, and $\sin x \neq o(x)$ as $x \to 0$.
- $\sin(x) = x - \frac{x^3}{6} + O(x^5)$ as $x \to 0$
2. Derivative

2.1. Set-up. $f(x)$ is a real-valued function defined on some neighborhood of $a$.

2.2. Definition. We say that $f$ is differentiable at $a$ provided the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists (and is finite). If that is the case, we denote the limit by $f'(a)$.

Note: by definition, a necessary condition for $f$ to be differentiable at $a$ is that $f$ is continuous at $a$.

2.3. If $f$ is differentiable at $a$, we can then re-write the limit as

$$\lim_{x \to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} = 0$$

which is the same as

$$f(x) = f(a) + f'(a)(x - x_0) + o(x - a)$$

2.4. Proposition. The following statements are true:

i) $f$ is differentiable at $a$.

ii) There exists a real number $k > 0$ such that $f(x) = f(a) + k(x - a) + o(x - a)$ as $x \to a$.

Proof. i $\Rightarrow$ ii. Take $k = f'(a)$.

ii $\Rightarrow$ i. Assume $f(x) = f(a) + k(x - a) + o(x - a)$ as $x \to a$. This simply means that $f(x) - f(a) - k(x - a) = o(x - a)$, which is equivalent to

$$\lim_{x \to a} \frac{f(x) - f(a) - k(x - a)}{x - a} = 0$$

But we can re-write the limit as

$$\lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} \right) - k = 0 \iff \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = k$$

Thus $f$ is differentiable at $a$ and $f'(a) = k$.

Proposition. If $f$ is differentiable at $a$, then $f(x)$ is continuous at $a$.

First proof: $\lim_{x \to a, x \neq a} f(x) - f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) = k \cdot 0 = 0$.

Second proof: $f(x) = f(a) + f'(a)(x - a) + o(x - a) = f(a) + O(x - a)$, hence $\lim_{x \to a} f(x) = f(a)$.

2.5. Linear approximation. Assume $f$ is continuous at $a$. For an arbitrary $k \in \mathbb{R}$, define $T_k(x) = f(a) + k(x - a)$. Then $T_k(x)$ is a linear function which takes the same value at $a$ as $f$: $T_k(a) = f(a)$ (draw the graph of $T_k$, a line). Since both $T_k$ and $f$ are continuous at $a$, we can write $f(x) = T_k(x) + o(1)$ as $x \to x_0$. We might want to conclude that $T_k$ provides a good enough approximation for $f$ near $a$ (after all, the difference $f - T_k$ does go to $0$ as $x \to a$). However, if we draw the graphs of $f$ and $T_k$ (with $k$ chosen ”arbitrary”) we see that aside from passing through the same point, the two curves don’t have much else in common. So the question is: is there a certain $k$ for which the difference $f(x) - T_k(x)$ (as $x \to a$) is even smaller than $o(1)$?

The first natural ”quantity” smaller than $o(1)$ is $O(x - a)$, but that is not good enough (let’s not wonder why). The next candidate would be $o(x - a)$. Does there exists a $k$ such that the corresponding $T_k$ approximates $f$ to $o(x - a)$ as $x \to a$?

From the discussion of the previous section we see that the answer is yes, provided $f$ is differentiable at $a$. If such a $k$ exists, then it is unique: $k = f'(a)$. Moreover, the graph of the corresponding $T_k$ is the tangent line at $a$ to the graph of $f$.

Remark: we now have the equation of the tangent line of a function. Being the graph of $T_k$ with $k = f'(a)$, it has equation $Y = f'(a)(X - a)$. 