2) Assuming the Fundamental Thm of Algebra, prove the following: If f and g are polynomials over the field of complex numbers, then \(\text{gcd}(f, g) = 1\) iff f and g have no common root.

The Fund Thm of Algebra states that \(\mathbb{C}\) is algebraically closed, i.e. every prime polynomial over \(\mathbb{C}\) has degree 1.

\[
g.c.d(f, g) = 1
\]

\[
\iff f = c_0 (x-c_1)^{n_1} (x-c_2)^{n_2} \ldots (x-c_n)^{n_n}
\]

\[
g = b_0 (x-b_1)^{m_1} (x-b_2)^{m_2} \ldots (x-b_m)^{m_m}
\]

These are the unique prime factorizations of f and g, and all of the \(c_i\) and \(b_j\) are distinct.

\[
\iff f \text{ and } g \text{ have no common root}
\]

4) Let A be a 2x2 matrix over a field F. Show that A is invertible iff \(\det A \neq 0\). When A is invertible, give a formula for \(A^{-1}\).

A is invertible \(\iff\) A equivalent to Identity Matrix

Let \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} 1 & b/a \\ 0 & \frac{-bc + d}{a} \end{bmatrix} \)

(Note: We can assume \(a \neq 0\). Otherwise switch the rows. If both \(a\) and \(c\) are zero, matrix not invertible anyway and \(\det A = 0\).)

Now, this matrix is equivalent to the identity matrix iff \(\frac{ad - bc}{a} \neq 0\), i.e. \(ad - bc = \det A \neq 0\).
continued

to find a formula for \( A^{-1} \):

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  b/a & 1/a \\
  c & d
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  b/a & 1/a & 0 \\
  0 & ad-bc & -c/a & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & 0 & d & -b \\
  0 & 1 & ad-bc & a
\end{bmatrix}
\]

so, \( A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \)

Based on this formula, it should make sense that \( A \) is invertible iff \( \det A \neq 0 \).

Let \( A \) be a 2x2 matrix over a field \( F \), and suppose that \( A^2 = 0 \).
Show for each scalar \( c_0 \) that \( \det (c_0 I - A) = c_0^2 \).

Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \)

\[ A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \]

Then:
1. \( aa + bc = 0 \)
2. \( ab + bd = b(a+d) = 0 \)
3. \( ac + dc = c(a+d) = 0 \)
4. \( bc + dd = 0 \)

\[ C_0 I - A = \begin{bmatrix} c_0 & 0 \\ 0 & c_0 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c_0-a & -b \\ -c & c_0-d \end{bmatrix} \]

\[ \det (c_0 I - A) = (c_0-a)(c_0-d) - bc \]
\[ = c_0^2 - c_0(a+d) + ad-bc \]

We need to show \( a+d \) and \( ad-bc \) are zero.

Well, since \( A^2 = 0 \), \( A \) cannot be invertible, so
\[ \det A = ad-bc = 0 \]

From properties 1, 2, 3, and 4, we see that \( a+d = 0 \)

Thus, \( \det (c_0 I - A) = c_0^2 \)
6. Let $K$ be a commutative ring with identity and $D$ an alternating $n$-linear function on $n \times n$ matrices over $K$. Show that

(a) $D(A) = 0$ if one of the rows of $A$ is 0.

(b) $D(B) = D(A)$ if $B$ is obtained from $A$ by adding a scalar multiple of one row of $A$ to another.

**Proof of (b):** Let $\alpha_1, \ldots, \alpha_n$ be the rows of $A$.

Then, without loss of generality, $\alpha_1, c\alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_n$ are the rows of $B$.

$$D(B) = D(\alpha_1, c\alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_n)$$

Since $D$ is $n$-linear,

$$D(B) = D(\alpha_1, c\alpha_1, \alpha_3, \ldots, \alpha_n) + D(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$$

$$= cD(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) + D(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$= 0 + D(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

since $D$ is alternating (i.e., 2 rows were equal)

$$= D(A)$$

Now, $\textbf{B} \Rightarrow \textbf{A}$:

A can be obtained from a matrix $C$ with 2 equal rows by subtracting these rows

Then, $0 = D(C) = D(A)$

since $D$ is alternating

7.162-163

4. An $n \times n$ matrix $A$ over a field $F$ is called **orthogonal** if $AA^T = I$. If $A$ is orthogonal, show that $\det A = \pm 1$. Give an example of an orthogonal matrix for which $\det A = -1$.

Well, $AA^T = I$

so $\det (AA^T) = \det I$

$(\det A)(\det A^T) = 1$
(4) continued

\[
\begin{align*}
\det(A^T \det A) &= 1 \quad \text{(since } \det(A^T) = \det(A)) \\
(\det A)^2 &= 1 \\
\Rightarrow \det A &= \pm 1.
\end{align*}
\]

Example:

\[
A = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

Then \(A^T = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}\)

so \(AA^T = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = I\), so \(A\) is orthogonal.

and \(\det A = -1 \checkmark\)

(5) An \(n\times n\) matrix \(A\) over the field of complex numbers is said to be \underline{unitary if} \(AA^* = I\) (where \(A^*\) is the conjugate transpose of \(A\)). If \(A\) is unitary, show that \(|\det A| = 1| \)

Let's first do the 2x2 case:

Let \(A = \begin{bmatrix}
a_1 + a_2 i & b_1 + b_2 i \\
c_1 + c_2 i & d_1 + d_2 i
\end{bmatrix}\)

Then \(A^* = \begin{bmatrix}
a_1 - a_2 i & c_1 - c_2 i \\
b_1 - b_2 i & d_1 - d_2 i
\end{bmatrix}\)

\[
\det A = \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} - \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} i = (a_1d_1 - b_1c_1 - a_2d_2 + b_2c_2) + i(a_1d_2 + a_2d_1 - b_1c_2 - b_2c_1)
\]

\[
\det A^* = (a_1d_1 - b_1c_1 - a_2d_2 + b_2c_2) + i(-a_1d_2 - a_2d_1 + b_1c_2 + b_2c_1)
\]

\[= \sqrt{P^2 + Q^2} \]

Now, \(\det(AA^*) = \det I\)

\(\det A)(\det A^*) = 1\)

\((|P + Q|)(|P - Q|)) = 1\)

\[P^2 + Q^2 = 1 \quad (\text{Recall, } |P + Q| = \sqrt{P^2 + Q^2})\]

\[|\det A| = 1 \checkmark\]
In a similar manner, for an \( n \times n \) unitary matrix, we'll end up with
\[
\det A = P + Qi \quad \text{and} \quad \det A^* = P - Qi \quad (\text{i.e. identical real parts, but negative imaginary parts})
\]
and
\[
(det A)(det A^*) = 1
\]
\[
(P + Qi)(P - Qi) = 1
\]
\[
P^2 + Q^2 = 1
\]
\[
|\det A|^2 = 1
\]
\[
|\det A| = 1
\]

Let \( A \) be an \( n \times n \) matrix over \( K \), a commutative ring with identity. Suppose \( A \) has the block form:
\[
A = \begin{bmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_k
\end{bmatrix}
\]

where \( A_j \) is an \( r_j \times r_j \) matrix.

Prove \( \det A = (\det A_1)(\det A_2) \cdots (\det A_k) \).

Recall the following: On p.157, the book proves that if we have an \( n \times n \) matrix of the block form \( \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \), where \( A \) is \( r \times r \), \( B \) is \( r \times s \), \( C \) is \( s \times s \),
then \( \det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = (\det A)(\det C) \).

Let's use this fact:
\[
\det A = \det \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix} = (\det A_1)\det \begin{bmatrix} A_2 & \cdots & 0 \\ 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}
\]
\[\text{in block form}\]
\[
= (\det A_1)(\det A_2) \cdots (\det A_k)
\]
Let $V$ be the vector space of all $n \times n$ matrices over the field $F$. Let $B$ be a fixed element of $V$ and let $T_B$ be the linear operator on $V$ defined by $T_B(A) = AB - BA$. Show that $\det T_B = 0$.

Note that the determinant of $T_B$ is the determinant of any matrix which represents $T_B$ in an ordered basis for $V$.

Next, by Thm 4 on p.160, a square matrix over a field is invertible iff the determinant is non-zero.

Thus, it suffices to show that $T_B$ is not invertible. To be invertible, the transformation must be 1-1 and onto, so if one of these does not hold, we're good to go.

Well, $T_B$ is not 1-1 since the kernel of $T_B$ is not just zero:

$$T_B(I) = IB - BI = B - B = 0$$
$$T_B(0) = 0B - B0 = 0$$

Hence, the determinant of $T_B$ is zero. ✓