Homework 5 Solutions

Due on 10/16/2001

Exercises 13.5 Page 810 - 811
10*, 12*, 16*, 18*

* means this question is graded by TAs and included in this solution.
^ means this question is included in this solution.
13.5.10. A particle moves so that \( \mathbf{r}(t) = 2 \cos 2t \mathbf{i} + 3 \cos t \mathbf{j} \).
(a) Show that the particles oscillates on an arc of the parabola
\[ 4y^2 - 9x = 18. \]
(b) Draw the path.
(c) What are the acceleration vectors at the points of zero velocity?
(d) Draw these vectors at the points in question.

[Solution]
(a) Let \( x = 2 \cos 2t \) and \( y = 3 \cos t \). We have
\[
4y^2 - 9x = 4(3 \cos t)^2 - 9(2 \cos 2t) \\
= 4(9 \cos^2 t) - 9(2(2 \cos^2 t - 1)) \\
= 36 \cos^2 t - 36 \cos^2 t + 18 \\
= 18.
\]
Since
\[-1 \leq \cos 2t \leq 1\]
and
\[-1 \leq \cos t \leq 1, \]
where \( t \in \mathbb{R} \), we have
\[-2 \leq x(t) \leq 2\]
and
\[-3 \leq y(t) \leq 3.\]
And, the path \( \mathbf{r}(t) \) consists only of the bounded arc
\[ x = \frac{4}{9}y^2 - 2, \quad -3 \leq y \leq 3. \]
Therefore, the motion traces out this arc twice on every \( t \)-interval of length \( 2t \).
(b) Included in (d).
(c) Since
\[ \mathbf{r}(t) = 2 \cos 2t \mathbf{i} + 3 \cos t \mathbf{j}, \]
we have
\[ \mathbf{r}'(t) = -4 \sin 2t \mathbf{i} - 3 \sin t \mathbf{j} \]
and
\[ \mathbf{r}''(t) = -8 \cos 2t \mathbf{i} - 3 \cos t \mathbf{j}. \]
At the points of zero velocity, we have
\[ \mathbf{r}'(t) = -4 \sin 2t \mathbf{i} - 3 \sin t \mathbf{j} = 0, \]
that is
\[
\begin{align*}
-4 \sin 2t &= 0 \\
-3 \sin t &= 0
\end{align*}
\]
which occurs at \( t = n\pi, \; n \in \mathbb{N} \). At such points, the acceleration is
\[
\begin{align*}
-8i - 3j, & \text{ if } n \text{ is even.} \\
-8i + 3j, & \text{ if } n \text{ is odd.}
\end{align*}
\]
13.5.12. (a) An object moves so that

\[ \mathbf{r}(t) = a_1 e^{bt} \mathbf{i} + a_2 e^{bt} \mathbf{j} + a_3 e^{bt} \mathbf{k}. \]

Show that, if \( b > 0 \), the object experience a repelling central force.

(b) An object moves so that

\[ \mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + (\sin t + \cos t) \mathbf{k}. \]

Show that the object experience an attracting central force.

(c) Compute the angular momentum \( \mathbf{L}(t) \) for the motion in (b).

[Solution]

(a) Since

\[ \mathbf{r}(t) = a_1 e^{bt} \mathbf{i} + a_2 e^{bt} \mathbf{j} + a_3 e^{bt} \mathbf{k}, \]

we have

\[ \mathbf{r}''(t) = a_1 b^2 e^{bt} \mathbf{i} + a_2 b^2 e^{bt} \mathbf{j} + a_3 b^2 e^{bt} \mathbf{k} = b^2 \mathbf{r}(t). \]

So,

\[ \mathbf{F}(t) = m \mathbf{r}''(t) = mb^2 \mathbf{r}(t), \]

that is \( \mathbf{F}(t) \) is a central force. Since \( b > 0 \), we know that the force \( \mathbf{F}(t) \) have the same direction with \( \mathbf{r}(t) \). This means that the object experience a repelling central force.

(b) Since

\[ \mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + (\sin t + \cos t) \mathbf{k}, \]

we have

\[ \mathbf{r}''(t) = - \sin t \mathbf{i} - \cos t \mathbf{j} - (\sin t + \cos t) \mathbf{k}. \]

So,

\[ \mathbf{F}(t) = m \mathbf{r}''(t) = -m \mathbf{r}(t), \]

that is \( \mathbf{F}(t) \) is a central force. Moreover, the force \( \mathbf{F}(t) \) have the opposite direction with \( \mathbf{r}(t) \). This means that the object experience a repelling central force.

(c) Since

\[ \mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + (\sin t + \cos t) \mathbf{k}, \]

we have

\[ \mathbf{v}(t) = \mathbf{r}'(t) = \cos t \mathbf{i} - \sin t \mathbf{j} - (\cos t - \sin t) \mathbf{k}. \]
The angular momentum in (b) is

\[ \mathbf{L} = \mathbf{r}(t) \times m \mathbf{v}(t) = m(\sin t \mathbf{i} + \cos t \mathbf{j} + (\sin t + \cos t) \mathbf{k}) \times \\
(\cos t \mathbf{i} - \sin t \mathbf{j} + (\cos t - \sin t) \mathbf{k}) = m \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\sin t & \cos t & \sin t + \cos t \\
\cos t & -\sin t & \cos t - \sin t
\end{vmatrix} \\
= m \begin{pmatrix}
\cos t & \sin t + \cos t \\
-\sin t & \cos t - \sin t
\end{pmatrix} \mathbf{i} - \begin{pmatrix}
\sin t & \sin t + \cos t \\
\cos t & \cos t - \sin t
\end{pmatrix} \mathbf{j} + \begin{pmatrix}
\sin t & \cos t \\
\cos t & -\sin t
\end{pmatrix} \mathbf{k} = m(\mathbf{i} + \mathbf{j} - \mathbf{k}). \]
13.5.16. Show that for an object of constant velocity the angular momentum is constant.

[Proof 1]
By definition, the angular momentum is
\[ L = r(t) \times m v(t). \]
We have
\[ L' = r'(t) \times m v(t) + r(t) \times m v'(t) \]
\[ = m v(t) \times v(t) + r(t) \times m v'(t). \]
Since the velocity is a constant, \( v'(t) = 0 \). And, \( v(t) \times v(t) = 0 \).
Hence \( L' = 0 \). So, the angular momentum, \( L \), is constant.

[Proof 2]
Since the velocity is constant, we may assume that \( v(t) = a_1 i + a_2 j + a_3 k \).
Then, we have
\[ r(t) = (a_1 t + c_1)i + (a_2 t + c_2)j + (a_3 t + c_3)k. \]
So, the angular momentum is
\[ L = r(t) \times m v(t) \]
\[ = m((a_1 t + c_1)i + (a_2 t + c_2)j + (a_3 t + c_3)k) \times (a_1 i + a_2 j + a_3 k) \]
\[ = m \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ t + a_1 & a_2 & a_3 \end{vmatrix} \]
\[ = m \begin{pmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ t + a_1 & a_2 & a_3 \end{pmatrix} \]
\[ = m \begin{pmatrix} c_2 a_3 - c_3 a_2 & i - m(c_1 a_3 - c_3 a_1)j + m(c_1 a_2 - c_2 a_1)k \end{pmatrix} \]
This means that the angular momentum is constant.
13.5.18. If an object of mass $m$ moves with velocity $v(t)$ subject to a force $F(t)$, the scalar product

$$F(t) \cdot v(t)$$

is called the *power* (expended by the force) and the number

$$\frac{1}{2} m [v(t)]^2$$

is called the *kinetic energy* of the object. Show that the time rate of change of the kinetic energy of an object is the power expended on it:

$$\frac{d}{dt} \left( \frac{1}{2} m [v(t)]^2 \right) = F(t) \cdot v(t).$$

**Proof**

$$\frac{d}{dt} \left( \frac{1}{2} m [v(t)]^2 \right) = \frac{1}{2} m \frac{d}{dt} [v(t)]^2$$

$$= \frac{1}{2} m \left( 2v(t) \frac{d}{dt} v(t) \right)$$

$$= m \frac{d}{dt} v(t) \cdot v(t)$$

$$= ma(t) \cdot v(t)$$

$$= F(t) \cdot v(t).$$

So, the time rate of change of the kinetic energy of an object is the power expended on it. 

$\blacksquare$