Answers for the Lagrange multiplier quiz

Suppose \( g(x, y) = 3x^3 + 4y^3 \) and let \( f(x, y) = x^2 + y^2 \).

1. Find \( \nabla f \) and \( \nabla g \).

[Answer: \( \nabla f = (2x, 2y) \), and \( \nabla g = (9x^2, 12y^2) \).]

2. Find the points on the level curve \( g(x, y) = 12 \) which satisfy the equation

\[
\nabla g = \lambda \nabla f
\]

for some nonzero \( \lambda \), and calculate \( f \) in those cases. [Don’t bother to calculate \( \lambda \) unless you have to. HINT: \( 18^2 = 324 \), \( 18^3 = 5832 \).]

[Answer: The equation asserts that \( (9x^2, 12y^2) = \lambda (2x, 2y) \), or in other words that

\[
9x^2 = 2\lambda x, \quad 12y^2 = 2\lambda y
\]

Now \( x \) and \( y \) can’t both be zero, since \( 3x^3 + 4y^3 = 12 \). If \( x = 0 \) then \( 4y^3 = 12 \), so \( y = 3^{1/3} \) and

\[
f(0, 3^{1/3}) = 3^{2/3}.
\]

Similarly, if \( y = 0 \) then \( 3x^3 = 12 \), so \( x = 4^{1/3} \) and

\[
f(0, 4^{1/3}) = 4^{2/3}.
\]

If \( x \) and \( y \) are both nonzero, then

\[
x = \frac{2}{9} \lambda, \quad y = \frac{1}{6} \lambda,
\]

so

\[
3 \left( \frac{2}{9} \lambda \right)^3 + 4 \left( \frac{1}{6} \lambda \right)^3 \lambda^3 = \frac{25}{486} \lambda^3 = 12,
\]

so \( \lambda^3 = \frac{5832}{25} \). On the other hand for these values of \( x \) and \( y \) we have

\[
f(x, y) = \left( \frac{2}{9} \right)^2 \lambda^2 + \left( \frac{1}{6} \right)^2 \lambda^2 = \frac{25}{324} \lambda^2;
\]

plugging in the value for \( \lambda \) derived above, we get

\[
f = \frac{25}{324} \left( \frac{5832}{25} \right)^{2/3} = 25^{1/3} = 5^{2/3}.
\]