110.202 Calculus III

Homework 12 Solutions

Due on 12/5/2001

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17.9.8. Verify the divergence theorem on the unit cube \(0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\) for the following vector fields.

\[ \mathbf{v}(x,y,z) = x\mathbf{i} + xy\mathbf{j} + xyz\mathbf{k}. \]

**[Solution]**

Since we have six faces in the unit cube, the flux of the cube is the summation of all six fluxes of each face.

- For the face \(x = 0, 0 \leq y \leq 1, 0 \leq z \leq 1\), we have the outgoing normal vector \(\mathbf{n}\) is \(-\mathbf{i}\). So, \(\mathbf{v} \cdot \mathbf{n} = -x = 0\). Hence, the flux of this face is \(\iint_S (\mathbf{v} \cdot \mathbf{n})d\sigma = \int_0^1 \int_0^1 0 dydz = 0\).

- For the face \(x = 1, 0 \leq y \leq 1, 0 \leq z \leq 1\), we have the outgoing normal vector \(\mathbf{n}\) is \(\mathbf{i}\). So, \(\mathbf{v} \cdot \mathbf{n} = x = 1\). Hence, the flux of this face is \(\iint_S (\mathbf{v} \cdot \mathbf{n})d\sigma = \int_0^1 \int_0^1 1 dydz = 1\).

- For the face \(0 \leq x \leq 1, y = 0, 0 \leq z \leq 1\), we have the outgoing normal vector \(\mathbf{n}\) is \(-\mathbf{j}\). So, \(\mathbf{v} \cdot \mathbf{n} = -xy = 0\). Hence, the flux of this face is \(\iint_S (\mathbf{v} \cdot \mathbf{n})d\sigma = \int_0^1 \int_0^1 0 dxdz = 0\).

- For the face \(0 \leq x \leq 1, y = 1, 0 \leq z \leq 1\), we have the outgoing normal vector \(\mathbf{n}\) is \(\mathbf{j}\). So, \(\mathbf{v} \cdot \mathbf{n} = xy = x\). Hence, the flux of this face is \(\iint_S (\mathbf{v} \cdot \mathbf{n})d\sigma = \int_0^1 \int_0^1 x dxdz = \frac{1}{2}\).

- For the face \(0 \leq x \leq 1, 0 \leq y \leq 1, z = 0\), we have the outgoing normal vector \(\mathbf{n}\) is \(-\mathbf{k}\). So, \(\mathbf{v} \cdot \mathbf{n} = -xyz = 0\). Hence, the flux of this face is \(\iint_S (\mathbf{v} \cdot \mathbf{n})d\sigma = \int_0^1 \int_0^1 0 dxdy = 0\).

- For the face \(0 \leq x \leq 1, 0 \leq y \leq 1, z = 0\), we have the outgoing normal vector \(\mathbf{n}\) is \(\mathbf{k}\). So, \(\mathbf{v} \cdot \mathbf{n} = xyz = xy\). Hence, the flux of this face is \(\iint_S (\mathbf{v} \cdot \mathbf{n})d\sigma = \int_0^1 \int_0^1 xy dxdy = \frac{1}{4}\).

Therefore, the flux of the cube \(C\) is

\[ \iint_{\partial C} (\mathbf{v} \cdot \mathbf{n})d\sigma = 0 + 1 + 0 + \frac{1}{2} + 0 + \frac{1}{4} = \frac{7}{4}. \]

On the other hand,

\[ \iiint_C (\nabla \cdot \mathbf{v})dxdydz = \int_0^1 \int_0^1 \int_0^1 (1 + x + xy)dxdydz = \frac{7}{4}. \]

Thus, the divergence theorem holds. \(\blacksquare\)
17.10.14. The sphere \( x^2 + y^2 + z^2 = a^2 \) intersects the plane \( x + 2y + z = 0 \) in a curve \( C \). Calculate the circulation of \( \mathbf{v} = 2yi - zj + 2xk \) about \( C \) using Stokes’ theorem.

**[Solution]**

We have

\[
\nabla \times \mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2y & -z & 2x
\end{vmatrix} = (0 - (-1))i - (2 - 0)j + (0 - 2)k = i - 2j - 2k.
\]

Since the plane \( x + 2y + z = 0 \) passes through the origin, it intersects the sphere in a circle of radius \( a \). The surface \( S \) bounded by this circle is a disc of radius \( a \) with the area \( A = \pi a^2 \) which is a subset of the plane \( x + 2y + z = 0 \). So, the upper normal vector of the surface is the same with the plane \( x + 2y + z = 0 \), which is \((1, 2, 1)\). Thus, the unit upper normal vector of the surface is

\[\mathbf{n} = \frac{1}{\sqrt{6}}(i + 2j + k)\).

By using Stokes’ theorem, the circulation of \( \mathbf{v} \) about \( C \) with respect to \( \mathbf{n} \) is

\[
\oint_C \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = \iint_S [(\nabla \times \mathbf{v}) \cdot \mathbf{n}] d\sigma
\]

\[
= \iint_S (i - 2j - 2k) \cdot \left( \frac{1}{\sqrt{6}}(i + 2j + k) \right) d\sigma
\]

\[
= \iint_S \frac{-5}{\sqrt{6}} d\sigma
\]

\[
= -\frac{5}{\sqrt{6}} A
\]

\[
= -\frac{5}{\sqrt{6}} \pi a^2.
\]

If \(-\mathbf{n}\) is used, then the circulation is \(\frac{5}{\sqrt{6}} \pi a^2\).
17.10.16. The cylinder \( x^2 + y^2 = b^2 \) intersects the plane \( y + z = a^2 \) in a curve \( C \). Assume \( a^2 \geq b > 0 \). Calculate the circulation of \( v = xyi + yzj + xzk \) about \( C \) using Stokes’ theorem.

**[Solution]**

We have

\[
\nabla \times v = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
xy & yz & xz
\end{vmatrix}
= (0 - y)\hat{i} - (z - 0)\hat{j} + (0 - x)\hat{k}
= -yi - zj - xk.
\]

The curve \( C \) bounds a flat surface \( S \) that projects onto the disc \( x^2 + y^2 = b^2 \) in the \( xy \)-plane which is a subset of the plane \( y + z = a^2 \). So, the upper normal vector of the surface is the same with the plane \( y + z = a^2 \), which is \((0, 1, 1)\) with the length \( \sqrt{2} \). Thus, the unit upper normal vector of the surface is

\[
n = \frac{1}{\sqrt{2}}(\hat{j} + \hat{k}).
\]

**[Method I]**

Let \( \theta \) be the angle between \( n \) and \( \hat{k} \). We have \( \cos \theta = n \cdot \hat{k} = \frac{1}{\sqrt{2}} \).

Thus \( \sec \theta = \sqrt{2} \). Therefore, the area of \( S \) is \( \pi b^2 \sqrt{2} \). By using Stokes’ theorem, the circulation of \( v \) about \( C \) with respect to \( n \) is

\[
\oint_C v(r) \cdot dr = \iint_S [\nabla \times v] \cdot n \, d\sigma
= \iint_S (yi - zj - xk) \cdot \left( \frac{1}{\sqrt{2}}(j + k) \right) \, d\sigma
= -\frac{1}{\sqrt{2}} \iint_S (x + z) \, d\sigma.
\]

Since \( S \) is symmetric with respect to \( x \)-axis, we have \( \iiint_S x \, d\sigma = 0 \). And, by definition, we have \( \iiint_S z \, d\sigma = \bar{z}A \) where \( \bar{z} \) is the \( z \)-coordinate of the centroid of \( S \). Moreover, by symmetry of \( S \), we
have \( z = a^2 \). Thus,

\[
\int_C \mathbf{v}(r) \cdot dr = -\frac{1}{\sqrt{2}} \iint_s (x + z) d\sigma
\]

\[
= -\frac{1}{\sqrt{2}} (0 + a^2 A)
\]

\[
= -\pi a^2 b^2.
\]

If \(-\mathbf{n}\) is used, then the circulation is \( \pi a^2 b^2 \).

[Method II]

Since \( S \) projects onto the disc \( x^2 + y^2 = b^2 \) in the \( xy \)-plane, we have a parametrization of this projection which is \( x = r \cos \theta, y = r \sin \theta \) where \( 0 \leq r \leq b, 0 \leq \theta \leq 2\pi \). Moreover, \( S \) is a subset of the plane \( y + z = a^2 \). We have a parametrization of the \( z \)-coordinate of the point in \( S \) which is \( z = a^2 - y = a^2 - r \sin \theta \). Hence, the parametrization of \( S \) is \( R(r, \theta) = (r \cos \theta, r \sin \theta, a^2 - r \sin \theta) \).

Moreover,

\[
\mathbf{N}(r, \theta) = R'_r \times R'_\theta
\]

\[
= \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos \theta & \sin \theta & -\sin \theta \\
-r \sin \theta & \cos \theta & -r \cos \theta
\end{vmatrix}
\]

\[
= (\sin \theta (-r \cos \theta) - (\sin \theta) r \cos \theta) \mathbf{i}
\]

\[
- (\cos \theta (-r \cos \theta) - (\sin \theta)(-r \sin \theta)) \mathbf{j}
\]

\[
+ (\cos \theta (r \cos \theta) - (\sin \theta)(-r \sin \theta)) \mathbf{k}
\]

\[
= 0 \mathbf{i} + r \mathbf{j} + r \mathbf{k}.
\]

Thus,

\[
||\mathbf{N}(r, \theta)|| = \sqrt{r^2 + r^2} = \sqrt{2r}.
\]

By using Stokes' theorem, the circulation of \( \mathbf{v} \) about \( C \) with respect to \( \mathbf{n} \) is

\[
\oint_C \mathbf{v}(r) \cdot dr = \iint_S [\nabla \times \mathbf{v}] \cdot \mathbf{n} \, d\sigma
\]

\[
= -\frac{1}{\sqrt{2}} \iint_s (x + z) d\sigma.
\]
By Formula 17.7.2, we have
\[
\iint_S (x + z) \, d\sigma = \int_0^b \int_0^{2\pi} (r \cos \theta + a^2 - r \sin \theta) \sqrt{2r} \, d\theta \, dr
\]
\[
= \sqrt{2} \int_0^b \int_0^{2\pi} [r^2 \sin \theta + a^2 r \theta + r^2 \cos \theta] \, dr
\]
\[
= \sqrt{2} \int_0^b a^2 2\pi r \, dr
\]
\[
= \sqrt{2} [a^2 \pi r^2]_0^b
\]
\[
= \sqrt{2} a^2 b^2 \pi.
\]
Hence,
\[
\oint_C \mathbf{v}(r) \cdot d\mathbf{r} = -\frac{1}{\sqrt{2}} \iint_S (x + z) \, d\sigma = -a^2 b^2 \pi.
\]