5.4. #10 Consider a consistent system $A\vec{x} = \vec{b}$.

(a) Show that this system has a solution $\vec{x}_0$ in $(\ker A)^\perp$. *Hint:* An arbitrary solution $\vec{x}$ of the system can be written as $\vec{x} = \vec{x}_h + \vec{x}_0$, where $\vec{x}_h$ is in $\ker (A)$ and $\vec{x}_0$ is in $(\ker A)^\perp$.

(b) Show that the system $A\vec{x} = \vec{b}$ has only one solution in $(\ker A)^\perp$. *Hint:* If $\vec{x}_0$ and $\vec{x}_1$ are two solutions in $(\ker A)^\perp$, think about $\vec{x}_1 - \vec{x}_0$.

(c) If $\vec{x}_0$ is the solution in $(\ker A)^\perp$ and $\vec{x}_1$ is another solution of the system $A\vec{x} = \vec{b}$, show that $\|\vec{x}_0\| < \|\vec{x}_1\|$. The vector $\vec{x}_0$ is called the *minimal solution* of the linear system $A\vec{x} = \vec{b}$.

**[Solution]**

Since $A\vec{x} = \vec{b}$ is a consistent system, this system has at least one solution, namely $\vec{x}$.

1. (a) Define $\vec{x}_h = \text{proj}_{\ker (A)} \vec{x} \in \ker (A)$. Define $\vec{x}_0 = \vec{x} - \vec{x}_h$. Then $A\vec{x}_0 = A(\vec{x} - \vec{x}_h) = A\vec{x} - A\vec{x}_h = \vec{b} - \vec{0} = \vec{b}$. Hence, $\vec{x}_0$ is also a solution of the system $A\vec{x} = \vec{b}$. Assume that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ form an orthonormal basis of $\ker (A)$. Then we have $\vec{x}_h = \text{proj}_{\ker (A)} \vec{x} = (\vec{v}_1 \cdot \vec{x}) \vec{v}_1 + (\vec{v}_2 \cdot \vec{x}) \vec{v}_2 + \cdots + (\vec{v}_n \cdot \vec{x}) \vec{v}_n$. For all $i = 1, 2, \ldots, n$, $\vec{x}_i = (\vec{x} - \vec{x}_h) \cdot \vec{v}_i = \vec{x} \cdot \vec{v}_i - \vec{x}_h \cdot \vec{v}_i = \vec{x} \cdot \vec{v}_i - (0 + \cdots + 0 + \vec{x} \cdot \vec{v}_i + 0 + \cdots + 0) = 0$. This implies that $\vec{x}_i \cdot \vec{y} = 0$ for all $\vec{y} \in \ker (A)$. So, $\vec{x}_0 \in (\ker A)^\perp$. Thus, $A\vec{x} = \vec{b}$ has a solution $\vec{x}_0$ in $(\ker A)^\perp$.

(b) Assume that $\vec{x}_0$ and $\vec{x}_1$ are two solutions in $(\ker A)^\perp$. Thus, we have $A\vec{x}_0 = \vec{b}$ and $A\vec{x}_1 = \vec{b}$. So, $A(\vec{x}_1 - \vec{x}_0) = A\vec{x}_1 - A\vec{x}_0 = \vec{b} - \vec{b} = \vec{0}$. This implies that $\vec{x}_1 - \vec{x}_0 \in \ker (A)$. Moreover, since $(\ker A)^\perp$ is a linear space, we have $\vec{x}_1 - \vec{x}_0 \in (\ker A)^\perp$. By Fact 5.4.2(c) in the textbook, we have $(\ker A)^\perp \cap \ker (A) = \{\vec{0}\}$. Therefore, $\vec{x}_1 - \vec{x}_0 \in (\ker A)^\perp \cap \ker (A) = \{\vec{0}\}$, that is, $\vec{x}_1 = \vec{x}_0$. Hence, the system $A\vec{x} = \vec{b}$ has only one solution in $(\ker A)^\perp$.

(c) Since $\vec{x}_0$ and $\vec{x}_1$ are different solutions of the system $A\vec{x} = \vec{b}$, we have $\vec{x}_1 - \vec{x}_0 \neq \vec{0}$. Define $\vec{x} = \vec{x}_1 - \vec{x}_0$. We have $A\vec{x} = A(\vec{x}_1 - \vec{x}_0) = A\vec{x}_1 - A\vec{x}_0 = \vec{b} - \vec{b} = \vec{0}$. This implies $\vec{x} \in \ker (A)$. Therefore, $\vec{x}_0 \perp \vec{x}$. Since $\vec{x}_0 \perp \vec{x}$, we know that, by Pythagorean Theorem, $\|\vec{x}_1\| = \|\vec{x}_1 - \vec{x}_0 + \vec{x}_0\| = \|\vec{x}_1 - \vec{x}_0\| + \|\vec{x}_0\|$. Hence, we have $\|\vec{x}_0\| < \|\vec{x}_1\|$ since $\vec{x}_1 - \vec{x}_0 \neq \vec{0}$.

5.4. #12 Using Exercise 10 as a guide, define the term "minimal least-squares solution" of a linear system. Explain why the minimal least-squares solution $\vec{x}_*$ of a linear system $A\vec{x} = \vec{b}$ is in $(\ker A)^\perp$.

**[Solution]**

By Fact 5.4.6 in the textbook, the least-squares solutions of the linear system $A\vec{x} = \vec{b}$ are the exact solutions of the consistent system $A^TA\vec{x} = A^T\vec{b}$. The minimal solution of this normal equation in the sense of Exercise 10 is called the minimal least-squares solution of the system $A\vec{x} = \vec{b}$. Equivalently, the minimal least-squares solution of $A\vec{x} = \vec{b}$ can be defined as the minimal solution of the consistent system $A\vec{x} = \text{proj}_{\ker (A)} \vec{b}$. 
