110.202. Calculus III
2004 Summer
Quiz II
7/20/2004 12:00 Noon

1. Evaluate

\[ \int_0^1 \int_0^{\arcsin y} y \cos xy \, dx \, dy. \]

[Solution]

\[ \int_0^1 \int_0^{\arcsin y} y \cos xy \, dx \, dy = \int_0^1 \left[ \int_0^{\arcsin y} y \cos xy \, dx \right] \, dy \]

\[ = \int_0^1 \left[ (\sin xy) \bigg|_{x=0}^{\arcsin y} \right] \, dy \]

\[ = \int_0^1 [\sin (\arcsin y) - \sin 0] \, dy \]

\[ = \int_0^1 y \, dy \]

\[ = \frac{y^2}{2} \bigg|_{y=0}^{1} \]

\[ = \frac{1}{2} - \frac{0^2}{2} \]

\[ = \frac{1}{2} \]

2. Sketch the region of integration of the following integral:

\[ \int_1^4 \int_1^{\sqrt{x}} (x^2 + y^2) \, dy \, dx, \]

interchange the order, and evaluate.
According to the region, we have

\[
\int_1^4 \int_1^{\sqrt{x}} (x^2 + y^2) \, dy \, dx
\]
\[
= \int_1^2 \int_y^4 (x^2 + y^2) \, dx \, dy
\]
\[
= \int_1^2 \left[ \left( \frac{x^3}{3} + xy^2 \right) \right]_{x=y^2}^4 \, dy
\]
\[
= \int_1^2 \left[ \left( \frac{4^3}{3} + 4y^2 \right) \right] dy
\]
\[
= \left( \frac{64}{3} y + \frac{4y^3}{3} - \frac{y^7}{21} - \frac{y^5}{5} \right)_{y=1}^2
\]
\[
= \frac{128}{3} + \frac{32}{3} - \frac{128}{21} - \frac{32}{5} - \frac{64}{3} - \frac{4}{3} + \frac{1}{21} + \frac{1}{5}
\]
\[
= \frac{1934}{105}.
\]

3. Integrate

\[ f(x, y) = x^2 + 2xy^2 + 2 \]

where \( D \) is the region bounded by the graph of \( y = -x^2 + x \), the \( x \) axis, and the lines \( x = 0 \) and \( x = 2 \).

According to the region, we have to divide \( D \) into 2 regions where \( D_1 = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq -x^2 + x\} \) and \( D_2 = \{(x, y) \mid 1 \leq x \leq 2, -x^2 + x \leq y \leq 0\} \). So, we have

\[
\iint_D (x^2 + 2xy^2 + 2) \, dA
\]
\[
= \int_0^1 \int_{-x^2+x}^0 (x^2 + 2xy^2 + 2) \, dy \, dx
\]
\[
+ \int_1^2 \int_{-x^2+x}^0 (x^2 + 2xy^2 + 2) \, dy \, dx.
\]
The first one becomes
\[\int_0^1 \int_0^{-x^2+x} \left( x^2 + 2xy^2 + 2 \right) dydx = \int_0^1 \left[ x^2y + \frac{2xy^3}{3} + 2y \right]_{y=0}^{y=-x^2+x} dx = \int_0^1 \left[ -x^4 + x^3 + \frac{2x(-x^2+x)^3}{3} - 2x^2 + 2x \right] dx = \frac{27}{70}.\]

The second one becomes
\[\int_1^2 \int_0^{-x^2+x} \left( x^2 + 2xy^2 + 2 \right) dydx = \int_1^2 \left[ x^2y + \frac{2xy^3}{3} + 2y \right]_0^{y=-x^2+x} dx = \int_0^1 \left[ -x^4 + x^3 + \frac{2x(-x^2+x)^3}{3} - 2x^2 + 2x \right] dx = (-1) \left( -\frac{584}{105} - \frac{27}{70} \right) = \frac{1249}{210}.\]

Hence \(\iint_D \left( x^2 + 2xy^2 + 2 \right) dA = \frac{27}{70} + \frac{1249}{210} = \frac{19}{3}.\)

4. Show that
\[4\pi \leq \iint_D \left( x^2 + y^2 + 1 \right) dxdy \leq 20\pi,\]
where \(D\) is the disk of radius 2 centered at the origin.

[Solution]
The area of the disk of radius 2 is \(\pi 2^2 = 4\pi\). Moreover, for a point \((x, y)\) in \(D\), we have \(x^2 + y^2 \leq 2^2\). Therefore, \(0 \leq x^2 + y^2 \leq 4\), that is, \(1 \leq x^2 + y^2 + 1 \leq 5\). By the Mean Value theorem, there exists a point \((x_0, y_0)\) in \(D\) such that \(\iint_D (x^2 + y^2 + 1) dxdy = (x_0^2 + y_0^2 + 1) \cdot A(D)\) where \(A(D)\) is the area of \(D\). Then we have
\[4\pi = 1 \cdot A(D) \leq \iint_D (x^2 + y^2 + 1) dxdy \leq 5 \cdot A(D) = 20\pi.\]
5. Evaluate

\[
\int_0^1 \int_0^y \int_0^{\sqrt{xy}} \frac{x}{x^2 + z^2} \, dz \, dx \, dy.
\]

[Solution]

Dividing \( \frac{x}{x^2 + z^2} \) by \( x^2 \), the integrand becomes \( \frac{1}{1 + \left( \frac{z}{x} \right)^2} \). Let \( u = \frac{z}{x} \). So, \( x \, du = dz \). Therefore,

\[
\int_0^1 \int_0^y \int_0^{\sqrt{xy}} \frac{x}{x^2 + z^2} \, dz \, dx \, dy
= \int_0^1 \int_0^y \int_0^{\sqrt{xy}} \frac{1}{1 + u^2} \, du \, dx \, dy
= \int_0^1 \int_0^y \left[ \tan^{-1} u \right]_{u=0}^{\sqrt{xy}} \, dx \, dy
= \int_0^1 \int_0^y \frac{\pi}{6} \, dx \, dy
= \int_0^1 \frac{\pi}{6} y \, dy
= \frac{\pi}{6} \left( \frac{y^2}{2} \right)_{y=0}^{1}
= \frac{\pi}{12}.
\]