1. Evaluate
\[ \int_0^1 \int_0^y \int_0^{\sqrt[4]{x}} \frac{z}{x^2 + z^2} \, dz \, dx \, dy. \]

[Solution]
Dividing \( \frac{x}{x^2 + z^2} \) by \( x^2 \), the integrand becomes \( \frac{1}{1 + \frac{z}{x}} \). Let \( u = \frac{z}{x} \). So, \( x \, du = dz \). Therefore,
\[
\int_0^1 \int_0^y \int_0^{\sqrt[4]{x}} \frac{x}{x^2 + z^2} \, dz \, dx \, dy \\
= \int_0^1 \int_0^y \int_0^{\sqrt[4]{x}} \frac{1}{1 + u^2} \, du \, dx \, dy \\
= \int_0^1 \int_0^y \left[ \tan^{-1} u \right]_{u=0}^{1} \, dx \, dy \\
= \int_0^1 \int_0^y \frac{\pi}{6} \, dx \, dy \\
= \int_0^1 \frac{\pi}{6} y \, dy \\
= \frac{\pi}{6} \left( y^2 \right)_{y=0}^{1} \\
= \frac{\pi}{12}.
\]

2. Evaluate
\[ \int_0^1 \int_{\tan^{-1} y}^{\pi} (\sec^5 x) \, dx \, dy. \]

[Solution]
By drawing the graph of the region, we change the order of integration into

\[
\int_0^1 \int_{\tan^{-1} y}^{\pi} (\sec^5 x) \, dx \, dy = \int_0^\pi \int_0^{\tan x} (\sec^5 x) \, dy \, dx
\]

\[
= \int_0^\pi (\sec^5 x) y \bigg|_{y=0}^{\tan x} \, dx
\]

\[
= \int_0^\pi (\sec^5 x) \tan x \, dx
\]

\[
= \int_1^{\sqrt{2}} (\sec^4 x) \, d \sec x
\]

\[
= \sec^5 x \bigg|_{\sec x=1}^{\sqrt{2}}
\]

\[
= \frac{(\sqrt{2})^5}{5} - \frac{1}{5}
\]

\[
= \frac{4\sqrt{2} - 1}{5}.
\]

3. Integrate \( f(x, y, z) = xyz \) along the path

\[ c(t) = (\cos t, \sin t, t) \]

where 0 ≤ t ≤ 2π.

[Solution]

We have \( c'(t) = (\sin t, \cos t, 1) \). Therefore, \( \|c'(t)\| = \sqrt{2} \).

Hence,

\[
\int_c f \, ds = \int_0^{2\pi} xyz \|c'(t)\| \, dt
\]

\[
= \int_0^{2\pi} (\cos t)(\sin t) \, t \sqrt{2} \, dt.
\]

Set \( u = t \) and \( dv = \cos t \sin t \, dt \). Then, we have \( du = dt \) and \( v = \frac{1}{2} \sin^2 t \). By integration by parts, we have

\[
\int_0^{2\pi} (\cos t)(\sin t) \, t \, dt = t \left( \frac{1}{2} \sin^2 t \right) \bigg|_{t=0}^{2\pi} - \int_0^{2\pi} \left( \frac{1}{2} \sin^2 t \right) \, dt
\]

\[
= -\frac{1}{2} \int_0^{2\pi} \left( \frac{1 - \cos 2t}{2} \right) \, dt
\]

\[
= -\frac{\pi}{2}.
\]
Therefore, 
\[
\int e f ds = \int_0^{2\pi} (\cos t) (\sin t) t \sqrt{2} dt = -\frac{\pi \sqrt{2}}{2}.
\]

4. Let $B$ be the region in the first quadrant bounded by the curves $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$ and $x^2 - y^2 = 4$. Evaluate

\[
\iiint_B (x^2 + y^2) \, dxdy
\]

using the change of variables $u = x^2 - y^2$ and $v = xy$.

[Solution]

Since $B$ is bounded by $x^2 - y^2 = 1$ and $x^2 - y^2 = 4$, we have $1 \leq u \leq 4$. Since $B$ is bounded by $xy = 1$ and $xy = 3$, we have $1 \leq v \leq 3$. Let $B^*$ be the region in the $uv$ plane bounded by $1 \leq u \leq 4$ and $1 \leq v \leq 3$. Let $T(x,y) = (u,v) = (x^2 - y^2, xy)$: $B \to B^*$ be a transformation. We have the Jacobian of $T$ is

\[
\frac{\partial (u,v)}{\partial (x,y)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2 + y^2).
\]

By the change of variables theorem, we have

\[
\iint_B 1 \cdot [2 (x^2 + y^2)] \, dxdy = \iint_{B^*} 1dudv.
\]

Since $\iint_{B^*} 1dudv$ is the area of $B^*$, we have $\iint_{B^*} 1dudv = (4 - 1)(3 - 1) = 6$. Hence, we have

\[
2 \iint_B (x^2 + y^2) \, dxdy = \iint_B 1 \cdot [2 (x^2 + y^2)] \, dxdy = \iint_{B^*} 1dudv = 6.
\]

This implies that

\[
\iint_B (x^2 + y^2) \, dxdy = 3.
\]

5. Evaluate

\[
\iiint_D (x^2 + y^2 + z^2) \, xyz \, dxdydz
\]

over the sphere $D = \{(x,y,z) \mid x^2 + y^2 + z^2 \leq 2\}$.

[Solution]
Using spherical coordinates, let \( x = \rho \sin \phi \cos \theta \), \( y = \rho \sin \phi \sin \theta \) and \( z = \rho \cos \phi \) where \( 0 \leq \rho \leq 2 \), \( 0 \leq \phi \leq \pi \) and \( 0 \leq \theta \leq 2\pi \). The Jacobian is \( \rho^2 \sin \phi \). Therefore,

\[
\int\int\int_D (x^2 + y^2 + z^2) \, xyz \, dx \, dy \, dz
= \int_0^2 \int_0^\pi \int_0^{2\pi} \rho^2 \left( \rho \sin \phi \cos \theta \right) \left( \rho \sin \phi \sin \theta \right) \left( \rho \cos \phi \right) \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho
= \int_0^2 \int_0^\pi \int_0^{2\pi} \rho^7 \left( \cos \phi \sin^3 \phi \right) \left( \cos \theta \sin \theta \right) \, d\theta \, d\phi \, d\rho
= \left( \int_0^2 \rho^7 \, d\rho \right) \cdot \left( \int_0^\pi \cos \phi \sin^3 \phi \, d\phi \right) \cdot \left( \int_0^{2\pi} \cos \theta \sin \theta \, d\theta \right)
= \left( \frac{\rho^8}{8} \bigg|_{\rho=0} \right) \cdot \left( \sin^4 \phi \bigg|_{\phi=0} \right) \cdot \left( \sin^4 \theta \bigg|_{\theta=0}^{2\pi} \right)
= \frac{2^8}{8} \cdot 0 \cdot 0
= 0.
\]

6. Find the center of mass of the region between \( y = x^2 \) ans \( y = x \) if the density function is

\[
\delta (x, y) = x + y.
\]

[Solution]

By the formula of the center of mass, we have

\[
\bar{x} = \frac{\iint_D x \delta (x, y) \, dx \, dy}{\iint_D \delta (x, y) \, dx \, dy}
\]

and

\[
\bar{y} = \frac{\iint_D y \delta (x, y) \, dx \, dy}{\iint_D \delta (x, y) \, dx \, dy}.
\]
By drawing down the region $D$, the region $D$ can be described as $0 \leq x \leq 1$ and $x^2 \leq y \leq x$. Therefore,

\[
\iint_D \delta(x, y) \, dxdy = \int_0^1 \int_{x^2}^x (x + y) \, dydx
\]

\[
= \int_0^1 \left[ \left( xy + \frac{y^2}{2} \right) \right]_{y=x^2}^x \, dx
\]

\[
= \int_0^1 \left[ \left( x^3 + \frac{x^2}{2} \right) - \left( x^3 + \frac{x^4}{2} \right) \right] \, dx
\]

\[
= \left( \frac{x^3}{2} - \frac{x^4}{4} - \frac{x^5}{10} \right)_{x=0}^{x=1}
\]

\[
= \frac{18}{120},
\]

\[
\iint_D x\delta(x, y) \, dxdy = \int_0^1 \int_{x^2}^x x (x + y) \, dydx
\]

\[
= \int_0^1 \left[ \left( x^2y + \frac{xy^2}{2} \right) \right]_{y=x^2}^x \, dx
\]

\[
= \int_0^1 \left[ \left( x^3 + \frac{x^3}{2} \right) - \left( x^4 + \frac{x^5}{2} \right) \right] \, dx
\]

\[
= \left( \frac{3x^4}{8} - \frac{x^5}{5} - \frac{x^6}{12} \right)_{x=0}^{x=1}
\]

\[
= \frac{11}{120},
\]

and

\[
\iint_D y\delta(x, y) \, dxdy = \int_0^1 \int_{x^2}^x y (x + y) \, dydx
\]

\[
= \int_0^1 \left[ \left( \frac{xy^2}{2} + \frac{y^3}{3} \right) \right]_{y=x^2}^x \, dx
\]

\[
= \int_0^1 \left[ \left( \frac{x^3 + x^3}{2} \right) - \left( x^5 + \frac{x^6}{3} \right) \right] \, dx
\]

\[
= \left( \frac{5x^4}{24} - \frac{x^6}{12} - \frac{x^7}{21} \right)_{x=0}^{x=1}
\]

\[
= \frac{13}{168}.\]
Hence, we have $\bar{x} = \frac{11}{120} = \frac{11}{18}$ and $\bar{y} = \frac{13}{120} = \frac{65}{126}$.

7. Determine whether the integral
\[ \iint_D \frac{x + y}{x^2 + 2xy + y^2} \, dx \, dy \]
exists where $D = [0, 1] \times [0, 1]$. If it exists, compute its value.

[Solution]

The integrand can be simplified to $\frac{1}{x+y}$ when $x + y \neq 0$. This integral is improper whenever $x + y = 0$. In our region $D$, there is only one point $(0, 0)$ which satisfies $x + y = 0$. Let $D_\delta = [\delta, 1] \times [0, 1]$. We have $D_\delta \rightarrow D$ as $\delta \rightarrow 0$ and $\frac{1}{x+y}$ is well-defined and integrable in $D_\delta$. Therefore,

\[
\iint_D \frac{x + y}{x^2 + 2xy + y^2} \, dx \, dy = \lim_{\delta \to 0} \int_{\delta}^{1} \int_{0}^{1} \frac{1}{x + y} \, dy \, dx
\]

\[
= \lim_{\delta \to 0} \int_{\delta}^{1} \left[ (\ln |x + y|)_{y = 0}^{1} \right] \, dx
\]

\[
= \lim_{\delta \to 0} \int_{\delta}^{1} [\ln (x + 1) - \ln (x)] \, dx
\]

\[
= \lim_{\delta \to 0} \{[(x + 1) \ln (x + 1) - x \ln x - x]|_{x = \delta}\}
\]

\[
= \lim_{\delta \to 0} \{2 \ln 2 - (\delta + 1) \ln (\delta + 1) + \delta \ln \delta\}.
\]

Since $\lim_{\delta \to 0} \delta \ln \delta = \lim_{\delta \to 0} \frac{\ln \delta}{\delta} = \lim_{\delta \to 0} \frac{1}{\delta} = \lim_{\delta \to 0} -\delta = 0$, we have

\[
\iint_D \frac{x + y}{x^2 + 2xy + y^2} \, dx \, dy = 2 \ln 2.
\]

8. During an interleague game against the Philadelphia Phillies, Oakland A’s third basemen Eric Chavez was given a day off to get Mark McLemore’s big bat into the lineup. Overcome by the boredom of watching the game from the sidelines, Chavez decides to steal the Phillie Phanatic’s four-wheeler. At time $t = 0$, Chavez begins to speed away in the four-wheeler, following the path defined by

\[ x(t) = 3 - 5 \cos(\pi t) \]

and

\[ y(t) = 1 + 5 \sin(\pi t) \]
where \( t \geq 0 \). How far does Chavez travel during the first 5 seconds of his escape?

**[Solution]**

The Chavez’s getaway path is \( \mathbf{c}(t) = (x(t), y(t)) = (3 - 5 \cos(\pi t), 1 + 5 \sin(\pi t)) \) for \( t \geq 0 \). The length of the path after 5 seconds is

\[
L(\mathbf{c}) = \int_{0}^{5} \|\mathbf{c}'(t)\| \, dt
\]

\[
= \int_{0}^{5} \sqrt{[-5\pi \sin(\pi t)]^2 + [5\pi \cos(\pi t)]^2} \, dt
\]

\[
= \int_{0}^{5} 5\pi \, dt
\]

\[
= 25\pi.
\]