1. Compute the line integral
\[ \int_C (\sin \pi x) \, dy - (\cos \pi y) \, dz \]
where \( C \) is the line segment starting at \((1, 0, 0)\) and ending at \((0, 1, 0)\).

**Solution**

The equation of the line segment is
\[
\mathbf{r}(t) = (1, 0, 0) + t (-1, 1, 0) = (1 - t, t, 0)
\]
where \( 0 \leq t \leq 1 \). Let \( \mathbf{F}(x, y, z) = (0, \sin \pi x, -\cos \pi y) \). Then the line integral becomes
\[
\int_C (\sin \pi x) \, dy - (\cos \pi y) \, dz = \int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s}
\]
\[
= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
\]
\[
= \int_0^1 (0, \sin \pi (1 - t), -\cos \pi t) \cdot (-1, 1, 0) \, dt
\]
\[
= \int_0^1 \sin \pi (1 - t) \, dt
\]
\[
= \left. \frac{1}{\pi} \cos \pi (1 - t) \right|_{t=0}^{t=1}
\]
\[
= \frac{2}{\pi}.
\]
2. Find
\[ \int \int_S x^2 \, dS \]
where \( S \) is the part of the plane \( x = z \) inside the cylinder \( x^2 + y^2 = 1 \).

[Solution]
Use cylindrical coordinates for the cylinder. Let \( x = r \cos \theta \), \( y = r \sin \theta \) and \( z = z \) where \( 0 \leq r \leq 1 \) and \( 0 \leq \theta \leq 2\pi \). To get a parametrization \( \Phi \) of \( S \), we have \( x = z = r \cos \theta \). So, we have
\[ \Phi (r, \theta) = (r \cos \theta, r \sin \theta, r \cos \theta) \]
where \( 0 \leq r \leq 1 \) and \( 0 \leq \theta \leq 2\pi \).

We calculate
\[ T_u \times T_v = (\cos \theta i + \sin \theta j + \cos \theta k) \times (-r \sin \theta i + r \cos \theta j - r \sin \theta k) = -r i + r k. \]

So, we have
\[ \|T_u \times T_v\| = \sqrt{2} r. \]

Therefore,
\[ \int \int_S x^2 \, dS = \int_0^1 \int_0^{2\pi} (r \cos \theta)^2 \left(\sqrt{2} r \right) dr d\theta \]
\[ = \sqrt{2} \left(\int_0^{2\pi} \cos^2 \theta d\theta \right) \left(\int_0^1 r^3 dr \right) \]
\[ = \sqrt{2} \left(\frac{1}{2} \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta \right) \left(\frac{1}{4} \right) \]
\[ = \frac{\sqrt{2} \pi}{4}. \]

3. Consider the surface
\[ \Phi (u, v) = (u^2 \cos v, u^2 \sin v, u). \]

Find the equation of the tangent plane at \((u, v) = (1, 0)\).

[Solution]
First, we compute
\[ T_u = (2u \cos v, 2u \sin v, 1) \]
and
\[ T_v = (-u^2 \sin v, u^2 \cos v, 0). \]
Therefore, we have the normal vector

\[ \mathbf{n} = (\mathbf{T}_u \times \mathbf{T}_v) (1,0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u \cos v & 2u \sin v & 1 \\ -u^2 \sin v & u^2 \cos v & 0 \end{vmatrix}_{(u,v)=(1,0)} \]

\[ = (-u^2 \cos v) \mathbf{i} - (u^2 \sin v) \mathbf{j} + (2u^3) \mathbf{k} \bigg|_{(u,v)=(1,0)} \]

\[ = -\mathbf{i} + 2\mathbf{k}. \]

at \((1,0)\). Moreover, \(\Phi(1,0) = (1,0,1)\). Then we have the equation of the tangent plane at \((u,v) = (1,0)\) is

\[ (-1) \cdot (x - 1) + 0 \cdot (y - 0) + 2 \cdot (z - 1) = 0, \]

that is,

\[ 2z - x = 1. \]

4. Find the area of the surface defined by

\[ x + y + z = 1 \text{ and } x^2 + 2y^2 \leq 1. \]

[Solution]

Let \(z = 1 - x - y = g(x,y)\). Then our surface \(S\) is the graph of \(g(x,y)\) defined on \(D = \{x^2 + 2y^2 \leq 1\}\). The surface area is

\[ A(S) = \iint_D \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} \, dA \]

\[ = \iint_D \sqrt{(-1)^2 + (-1)^2 + 1} \, dxdy \]

\[ = \iint_D \sqrt{3} \, dxdy. \]

Use polar coordinate to parametrize \(D\) as \(x = r \cos \theta\) and \(y = \frac{r}{\sqrt{2}} \sin \theta\) where \(0 \leq r \leq 1\) and \(0 \leq \theta \leq 2\pi\). The Jacobian is \(\frac{r}{\sqrt{2}}\).
So, we have
\[
A(S) = \int \int_D \sqrt{3} \cdot r \, dr \, d\theta
\]
\[
= \int_0^{2\pi} \left[ \int_0^1 \sqrt{3} \cdot \frac{r}{\sqrt{2}} \, dr \, d\theta \right]
\]
\[
= \frac{\sqrt{3}}{\sqrt{2}} \int_0^{2\pi} \int_0^1 r \, dr \, d\theta
\]
\[
= \frac{\sqrt{3}}{\sqrt{2}} \cdot (\pi)
\]
\[
= \frac{\sqrt{6} \pi}{2}.
\]

5. Evaluate the line integral
\[
\int_C 2xy^2z^2 \, dx + 2x^2yz^2 \, dy + 2x^2y^2z \, dz
\]
where \( C \) is an oriented simple curve connecting \((1,1,1)\) to \((2,3,4)\).

[Solution]
Let \( F(x,y,z) = (2xy^2z^2, 2x^2yz^2, 2x^2y^2z) \). Let \( f(x,y,z) = (xyz)^2 \). It is easy to see that \( f \) is of class \( C^1 \). Then we have \( F = \nabla f \). By theorem, we have
\[
\int_C 2xy^2z^2 \, dx + 2x^2yz^2 \, dy + 2x^2y^2z \, dz
\]
\[
= \int_C \nabla f \cdot ds
\]
\[
= f(2,3,4) - f(1,1,1)
\]
\[
= (2 \cdot 3 \cdot 4)^2 - (1 \cdot 1 \cdot 1)^2
\]
\[
= 575.
\]