9.3 #6 Find \(a \cdot b\), where \(a = \langle s, 2s, 3s \rangle\) and \(b = \langle t, -t, 5t \rangle\).

[Solution]

\[
a \cdot b = (s)(t) + (2s)(-t) + (3s)(5t) = 12st.
\]

9.3 #8 Find \(a \cdot b\), where \(a = 4j - 3k\) and \(b = 2i + 4j + 6k\).

[Solution]

\[
a \cdot b = (0)(2) + (4)(4) + (-3)(6) = -2.
\]

9.3 #10 If \(u\) is a unit vector, find \(u \cdot v\) and \(u \cdot w\).

[Solution]

By the figure, the length of the diagonal is \(\sqrt{1^2 + 1^2} = \sqrt{2}\) since we have a unit square. Thus, \(|v| = \frac{\sqrt{2}}{2}\) and \(|w| = 1\). Since it is a unit square, the angle between \(v, u\) is \(45^\circ\) and the angle between \(v, w\) is \(90^\circ\). Therefore, we have

\[
\quad u \cdot v = |u||v| \cos 45^\circ = 1 \times \frac{\sqrt{2}}{2} \times \frac{1}{\sqrt{2}} = \frac{1}{2}
\]

and

\[
\quad u \cdot w = |u||w| \cos 90^\circ = 1 \times 1 \times 0 = 0.
\]

9.3 #20 For what values of \(b\) are the vectors \(\langle -6, b, 2 \rangle\) and \(\langle b, b^2, b \rangle\) orthogonal?

[Solution]

If \(\langle -6, b, 2 \rangle\) and \(\langle b, b^2, b \rangle\) are orthogonal, then \(\langle -6, b, 2 \rangle \cdot \langle b, b^2, b \rangle = 0\). This implies that \(b^3 - 4b = 0\). Thus, we have \(b = 0, 2\) and \(-2\).

9.3 #28 For the vectors \(a = \langle 1, 2 \rangle\) and \(b = \langle -4, 1 \rangle\), find orth_a b.

[Solution]

From exercise 27, we have \(\text{orth}_a b = b - \text{proj}_a b\). By the formula in the textbook page 655,

\[
\text{proj}_a b = \left( \frac{a \cdot b}{|a|^2} a \right) = \left( \frac{(1)(-4) + (2)(1)}{1^2 + 2^2} \right) \langle 1, 2 \rangle = \left( \frac{-2}{5} \right) \langle 1, 2 \rangle = \langle -\frac{2}{5}, -\frac{4}{5} \rangle.
\]
Thus,
\[ \text{orth}_a b = b - \text{proj}_a b = \langle -4, 1 \rangle - \left\langle \frac{-2}{5}, \frac{-4}{5} \right\rangle = \left\langle -\frac{18}{5}, 9 \right\rangle. \]

9.3 #30 Suppose that \( a \) and \( b \) are nonzero vectors.

(a) Under what circumstances is \( \text{comp}_a b = \text{comp}_b a \)?

(b) Under what circumstances is \( \text{proj}_a b = \text{proj}_b a \)?

[Solution]

(a) By the formula in the textbook page 655, \( \text{comp}_a b = \frac{a \cdot b}{|a|^2} a \) and \( \text{comp}_b a = \frac{b \cdot a}{|b|^2} b \). Assume that \( \text{comp}_a b = \text{comp}_b a \). This implies that \( \frac{a \cdot b}{|a|^2} = \frac{b \cdot a}{|b|^2} \), that is, \(|a| = |b|\) since \( a \cdot b \) is always equal to \( b \cdot a \).

(b) By the formula in the textbook page 655, \( \text{proj}_a b = \left( \frac{a \cdot b}{|a|^2} a \right) \) and \( \text{proj}_b a = \left( \frac{b \cdot a}{|b|^2} b \right) \). Assume that \( \text{proj}_a b = \text{proj}_b a \). This implies that \( \frac{a \cdot b}{|a|^2} = \frac{b \cdot a}{|b|^2} \), that is, \( \frac{1}{|a|^2} a = \frac{1}{|b|^2} b \) since \( a \cdot b \) is always equal to \( b \cdot a \). Because they are equal, their length are equal. This tells us that \( \frac{1}{|a|^2} a = \frac{1}{|b|^2} b \). Thus, \( \frac{|a|}{|a|^2} = \frac{|b|}{|b|^2} \), that is, \(|a| = |b|\). Moreover, we know that \( a = b \).

9.3 #34 A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 50 N. The handle of the wagon is held at an angle of 30° above the horizontal. How much work is done?

[Solution]

The work done is \( 100 \times 50 \times \cos 30° = 2500\sqrt{3} \).

9.3 #38 Find the angle between a diagonal of a cube and a diagonal of one of its faces.

[Solution]

By putting a cube in the first octant with one of its vertex is the origin, we have a diagonal of the cube can be represented by the vector \( \langle 1, 1, 1 \rangle \) and a diagonal of the face in \( xy \) plane can be represented by the vector \( \langle 1, 1, 0 \rangle \). Thus, we have \( \langle 1, 1, 1 \rangle \cdot \langle 1, 1, 0 \rangle = \langle 1, 1, 1 \rangle \cdot \langle 1, 1, 0 \rangle \cos \theta \), where \( \theta \) is the angle between a diagonal of our cube and a diagonal of one of its faces. This implies that \( 2 = \sqrt{3} \sqrt{2} \cos \theta \), that is, \( \cos \theta = \frac{2}{\sqrt{6}} \). So, \( \theta = \cos^{-1} \left( \frac{2}{\sqrt{6}} \right) = 0.61548 \).

9.3 #40 If \( c = |a| \mathbf{b} + |b| \mathbf{a} \), where \( a \), \( b \), and \( c \) are all nonzero vectors, show that \( c \) bisects the angle between \( a \) and \( b \).

[Solution]

Let \( \theta \) be the angle between \( a \) and \( c \). Then, we have \( a \cdot c = |a| |c| \cos \theta \). Thus,
\[
\cos \theta = \frac{a \cdot c}{|a| |c|} = \frac{a \cdot (|a| \mathbf{b} + |b| \mathbf{a})}{|a| |c|} = \frac{|a| a \cdot b + |b| a \cdot a}{|a| |c|} = \frac{|a| a \cdot b + |b| a}{|a| |c|} = \frac{a \cdot b + |b| |a|}{|c|}.
\]
Let \( \phi \) be the angle between \( \mathbf{b} \) and \( \mathbf{c} \). Then, we have
\[
\mathbf{b} \cdot \mathbf{c} = \frac{\mathbf{b} \cdot (|\mathbf{a}| \mathbf{b} + |\mathbf{b}| \mathbf{a})}{|\mathbf{b}| |\mathbf{c}|} = \frac{|\mathbf{a}| |\mathbf{b} \cdot \mathbf{b} + |\mathbf{b}| \mathbf{b} \cdot \mathbf{a}}{|\mathbf{b}| |\mathbf{c}|} = \frac{|\mathbf{a}| |\mathbf{b}| + |\mathbf{b}| \cdot \mathbf{a}}{|\mathbf{b}| |\mathbf{c}|}.
\]
Therefore, we have \( \cos \theta = \cos \phi \). Since our angles \( \theta \) and \( \phi \) are both between 0° and 180°, we have \( \theta = \phi \).

9.3 #44 The Triangle Inequality for vectors is
\[
|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|
\]
(a) Give a geometric interpretation of the Triangle Inequality.
(b) Use the Cauchy-Schwarz Inequality from Exercise 43 to prove the Triangle Inequality.

\[\text{Solution}\]
(a) The sum of the lengths of two sides of a triangle is greater than the length of the third side.
(b) Starts from
\[
|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + |\mathbf{b}|^2
\]
By using the Cauchy-Schwarz Inequality \( |\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}| |\mathbf{b}| \), we have
\[
|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + 2 |\mathbf{a} \cdot \mathbf{b}| + |\mathbf{b}|^2 \leq |\mathbf{a}|^2 + 2 |\mathbf{a} \cdot \mathbf{b}| + |\mathbf{b}|^2
\]
\[
\leq |\mathbf{a}|^2 + 2 |\mathbf{a}| |\mathbf{b}| + |\mathbf{b}|^2 = (|\mathbf{a}| + |\mathbf{b}|)^2.
\]
This implies that
\[
|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|.
\]

9.4 #4 The figure shows a vector \( \mathbf{a} \) in the \( xy \)-plane and a vector \( \mathbf{b} \) in the direction of \( \mathbf{k} \). Their length are \( |\mathbf{a}| = 3 \) and \( |\mathbf{b}| = 2 \).
(a) Find \( |\mathbf{a} \times \mathbf{b}| \).
(b) Use the right-hand rule to decide whether the components of \( \mathbf{a} \times \mathbf{b} \) are positive, negative, or 0.

\[\text{Solution}\]
(a) According to the figure, the angle between \( \mathbf{a} \) and \( \mathbf{b} \) is 90°. Thus,
\[
|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin 90° = 3 \times 2 \times 1 = 6.
\]
(b) Use the right-hand rule, we can see that \( \mathbf{a} \times \mathbf{b} \) should be sitting at \( xy \)-plane with positive \( x \) and negative \( y \). So, the \( x \)-components of \( \mathbf{a} \times \mathbf{b} \) is positive, the \( y \)-components of \( \mathbf{a} \times \mathbf{b} \) is negative and the \( z \)-components of \( \mathbf{a} \times \mathbf{b} \) is 0.

9.4 #6 Find the magnitude of the torque about \( P \) if a 36-lb force is applied as shown.

\[\text{Solution}\]
(Let us add a point \( Q \) in the end of our object in the figure at the page 664 – the point with all three colors, blue, red and dot-black.)
We have a vector \( \overrightarrow{PQ} \) which is the diagonal of a square with length \( \sqrt{4^2 + 4^2} = 4\sqrt{2} \). The angle between our force and \( \overrightarrow{PQ} \) is \( 45^\circ + 30^\circ = 75^\circ \). Thus, the torque is \( 4\sqrt{2} \times 36 \times \sin 75^\circ = 72 \left( 1 + \sqrt{3} \right) = 196.71 \).

9.4 #16 Find the area of the parallelogram with vertices \( K(1,2,3) \), \( L(1,3,6) \), \( M(3,8,6) \), and \( N(3,7,3) \).

[Solution] After plotting these four points in a coordinate system, we have a parallelogram \( KLMN \). The vector \( \overrightarrow{KL} = (1-1, 3-2, 6-3) = (0,1,3) \) and \( \overrightarrow{KN} = (3-1, 7-2, 3-3) = (2,5,0) \). Thus, the area of the parallelogram \( KLMN \) is

\[
\left| \overrightarrow{KL} \times \overrightarrow{KN} \right| = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 2 & 5 & 0 \end{vmatrix} \right| = \left| \begin{vmatrix} 1 & 3 & 0 \\ 5 & 0 & 3 \\ 2 & 0 & 2 \end{vmatrix} \right| \mathbf{i} + \left| \begin{vmatrix} 0 & 1 & 3 \\ 2 & 0 & 2 \\ 5 & 2 & 0 \end{vmatrix} \right| \mathbf{k} = \left| -15 \mathbf{i} + 6 \mathbf{j} - 2 \mathbf{k} \right| = \sqrt{\left(-15\right)^2 + 6^2 + (-2)^2} = \sqrt{265}.
\]

9.4 #20 Let \( \mathbf{v} = 5\mathbf{j} \) and let \( \mathbf{u} \) be a vector with length 3 that starts at the origin and rotates in the \( xy \)-plane. Find the maximum and minimum values of the length of the vector \( \mathbf{u} \times \mathbf{v} \). In what direction does \( \mathbf{u} \times \mathbf{v} \) point?

[Solution] Rotating a vector with length 3 that starts at the origin in the \( xy \)-plane forms a circle with center \((0,0)\) and radius 3. So, the coordinates of \( \mathbf{u} \) should look like \((x, \pm \sqrt{3^2 - x^2}, 0)\) where \( 0 \leq x \leq 3 \). Therefore,

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & \pm \sqrt{3^2 - x^2} & 0 \\ 0 & 5 & 0 \end{vmatrix} = 5x \mathbf{k},
\]

where \( 0 \leq x \leq 3 \). The length of \( \mathbf{u} \times \mathbf{v} \) is \( \sqrt{0^2 + 0^2 + (5x)^2} = 5x \). Thus, the maximum and minimum values of the length of the vector \( \mathbf{u} \times \mathbf{v} \) are 15 and 0, respectively. Also, \( \mathbf{u} \times \mathbf{v} \) point in the positive \( z \)-direction.

9.4 #28 (a) Let \( P \) be a point not on the plane that passes through the points \( Q \), \( R \), and \( S \). Show that the distance \( d \) from \( P \) to the plane is

\[
d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}.
\]
where \( \mathbf{a} = \overrightarrow{QR}, \mathbf{b} = \overrightarrow{QS}, \) and \( \mathbf{c} = \overrightarrow{QP}. \)

(b) Use the formula in part (a) to find the distance from the point \( P(2, 1, 4) \) to the plane through the points \( Q(1, 0, 0), R(0, 2, 0), \) and \( S(0, 0, 3). \)

**[Solution]**

(a) Consider a parallelepiped formed by \( \mathbf{a} = \overrightarrow{QR}, \mathbf{b} = \overrightarrow{QS}, \) and \( \mathbf{c} = \overrightarrow{QP}. \) (Refers to the Figure 7 in the textbook page 662.) The height of this parallelepiped is the distance \( d \) from \( P \) to the plane. The volume of this parallelepiped is \( |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| \). Also, the area of the parallelogram spanned by \( \mathbf{a} \) and \( \mathbf{b} \) is \( |\mathbf{a} \times \mathbf{b}|. \) Since \( V = Ah, \) we know that

\[
d = \frac{|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|}{|\mathbf{a} \times \mathbf{b}|}
\]

by the formula in the textbook page 662.

(b) We have

\[
\mathbf{a} = \overrightarrow{QR} = (0, 2, 0) - (1, 0, 0) = (-1, 2, 0),
\]

\[
\mathbf{b} = \overrightarrow{QS} = (0, 2, 0) - (0, 0, 3) = (0, 2, -3),
\]

and

\[
\mathbf{c} = \overrightarrow{QP} = (0, 2, 0) - (2, 1, 4) = (-2, 1, -4).
\]

So,

\[
|\mathbf{a} \times \mathbf{b}| = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  -1 & 2 & 0 \\
  0 & 2 & -3
\end{vmatrix}
= \begin{vmatrix}
  2 & 0 & \mathbf{i} \\
  0 & -3 & \mathbf{j} \\
 -1 & 0 & \mathbf{k}
\end{vmatrix}
= \begin{vmatrix}
  2 & 0 & -1 \\
  0 & 2 & -3 \\
 -1 & 0 & 2
\end{vmatrix}
= -6\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}
= \sqrt{(-6)^2 + (-3)^2 + (-2)^2}
= 7.
\]

Also,

\[
\mathbf{b} \times \mathbf{c} = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  0 & 2 & -3 \\
 -2 & 1 & -4
\end{vmatrix}
= \begin{vmatrix}
  2 & -3 & \mathbf{i} \\
  0 & -4 & \mathbf{j} \\
 -2 & 1 & \mathbf{k}
\end{vmatrix}
= -5\mathbf{i} + 6\mathbf{j} + 4\mathbf{k},
\]

and \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (-1, 2, 0) \cdot (-5, 6, 4) = 17. \) Therefore,

\[
d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|} = \frac{17}{7}.
\]

\[
9.4 \ #30 \ \text{Prove the following formula for the vector triple product:}
\]

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}.
\]

**[Solution]**
Let \( \mathbf{a} = (a_1, a_2, a_3) \), \( \mathbf{b} = (b_1, b_2, b_3) \) and \( \mathbf{c} = (c_1, c_2, c_3) \). Then,

\[
\mathbf{b} \times \mathbf{c} = \begin{vmatrix}
  i & j & k \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix} = \begin{vmatrix}
  b_2 & b_3 \\
  c_2 & c_3 \\
\end{vmatrix} i - \begin{vmatrix}
  b_1 & b_3 \\
  c_1 & c_3 \\
\end{vmatrix} j + \begin{vmatrix}
  b_1 & b_2 \\
  c_1 & c_2 \\
\end{vmatrix} k
\]

\[
= (b_2 c_3 - b_3 c_2) i - (b_1 c_3 - b_3 c_1) j + (b_1 c_2 - b_2 c_1) k. 
\]

Moreover,

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \\
= \begin{vmatrix}
  i & j & k \\
  a_1 & a_2 & a_3 \\
  b_2 c_3 - b_3 c_2 & b_3 c_1 - b_1 c_3 & b_1 c_2 - b_2 c_1 \\
\end{vmatrix} \\
= \begin{vmatrix}
  a_2 & a_3 \\
  b_3 c_1 - b_1 c_3 & b_1 c_2 - b_2 c_1 \\
\end{vmatrix} i - \begin{vmatrix}
  a_1 & a_3 \\
  b_2 c_3 - b_3 c_2 & b_1 c_2 - b_2 c_1 \\
\end{vmatrix} j + \begin{vmatrix}
  a_1 & a_2 \\
  b_2 c_3 - b_3 c_2 & b_3 c_1 - b_1 c_3 \\
\end{vmatrix} k \\
= [a_2 (b_1 c_2 - b_2 c_1) - a_3 (b_3 c_1 - b_1 c_3)] i - [a_1 (b_1 c_2 - b_2 c_1) - a_3 (b_2 c_3 - b_3 c_2)] j + [a_1 (b_3 c_1 - b_1 c_3) - a_2 (b_2 c_3 - b_3 c_2)] k \\
= [b_1 (a_2 c_2 + a_3 c_3) - (a_2 b_2 + a_3 b_3) c_1] i + [(a_1 c_1 + a_3 c_3) b_2 - (a_1 b_1 + a_3 b_3) c_2] j + [(a_2 c_2 + a_1 c_1) b_3 - (a_1 b_1 + a_2 b_2) c_3] k \\
= [(a_1 c_1 + a_2 c_2 + a_3 c_3) b_1 - (a_1 b_1 + a_2 b_2 + a_3 b_3) c_1] i + [(a_1 c_1 + a_2 c_2 + a_3 c_3) b_2 - (a_1 b_1 + a_2 b_2 + a_3 b_3) c_2] j + [(a_1 c_1 + a_2 c_2 + a_3 c_3) b_3 - (a_1 b_1 + a_2 b_2 + a_3 b_3) c_3] k \\
= [(a \cdot c) b_1 - (a \cdot b) c_1] i + [(a \cdot c) b_2 - (a \cdot b) c_2] j + [(a \cdot c) b_3 - (a \cdot b) c_3] k \\
= (a \cdot c) (b_1 i + b_2 j + b_3 k) - (a \cdot b) (c_1 i + c_2 j + c_3 k) \\
= (a \cdot c) \mathbf{b} - (a \cdot b) \mathbf{c}. 
\]

9.4 #32 Prove that

\[
\mathbf{a} \times \mathbf{b} \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix}
  \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\
  \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \\
\end{vmatrix}. 
\]

**[Solution]**

Note that

\[
\begin{vmatrix}
  \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\
  \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \\
\end{vmatrix} = (\mathbf{a} \cdot \mathbf{c}) (\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c}) (\mathbf{a} \cdot \mathbf{d}).
\]

Using Formula 6 in the textbook page 663, we have

\[
\mathbf{a} \times \mathbf{b} \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{a} \cdot (\mathbf{b} \times (\mathbf{c} \times \mathbf{d})).
\]

By the result of exercise 30, we have

\[
\mathbf{b} \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{b} \cdot \mathbf{d}) \mathbf{c} - (\mathbf{b} \cdot \mathbf{c}) \mathbf{d}.
\]
Thus,

\[(a \times b) \cdot (c \times d) = a \cdot ((b \cdot d)c - (b \cdot c)d) = (b \cdot d)(a \cdot c) - (b \cdot c)(a \cdot d) = \begin{vmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{vmatrix}.\]

9.4 #34 If \(v_1, v_2, \) and \(v_3\) are noncoplanar vectors, let

\[
k_1 = \frac{v_2 \times v_3}{v_1 \cdot (v_2 \times v_3)},
\]

\[
k_2 = \frac{v_3 \times v_1}{v_1 \cdot (v_2 \times v_3)},
\]

\[
k_3 = \frac{v_1 \times v_2}{v_1 \cdot (v_2 \times v_3)}.
\]

(a) Show that \(k_i\) is perpendicular to \(v_j\) if \(i \neq j\).

(b) Show that \(k_i \cdot v_i = 1\) for \(i = 1, 2, 3\).

(c) Show that \(k_1 \cdot (k_2 \times k_3) = \frac{1}{v_1 \cdot (v_2 \times v_3)}\).

[Solution]

(a) Note that \(v_j \times v_j = 0\) for \(j = 1, 2, 3\) and \(a \cdot (b \times c) = c \cdot (a \times b) = b \cdot (c \times a)\). So, if \(i \neq j\), it is easy to see that \(k_i \cdot v_j = 0\) since \(v_j\) happens to be in the numerator. Thus, \(k_i\) is perpendicular to \(v_j\) if \(i \neq j\).

(b) We can see that

\[k_1 \cdot v_1 = \left( \frac{v_2 \times v_3}{v_1 \cdot (v_2 \times v_3)} \right) \cdot v_1 = \frac{1}{v_1 \cdot (v_2 \times v_3)} [(v_2 \times v_3) \cdot v_1] = 1.\]

Similarly, \(k_2 \cdot v_2 = 1\) and \(k_3 \cdot v_3 = 1\).

(c) By the vector triple product, we have

\[
k_2 \times k_3 = \left( \frac{v_3 \times v_1}{v_1 \cdot (v_2 \times v_3)} \right) \times \left( \frac{v_1 \times v_2}{v_1 \cdot (v_2 \times v_3)} \right)
\]

\[= \left( \frac{1}{v_1 \cdot (v_2 \times v_3)} \right)^2 [(v_3 \times v_1) \times (v_1 \times v_2)]
\]

\[= \left( \frac{1}{v_1 \cdot (v_2 \times v_3)} \right)^2 [((v_3 \times v_1) \cdot v_2) v_1 - ((v_3 \times v_1) \cdot v_1) v_2]
\]

\[= \left( \frac{1}{v_1 \cdot (v_2 \times v_3)} \right)^2 [((v_1 \cdot (v_2 \times v_3)) v_1 - ((v_1 \cdot v_1) \cdot v_3) v_2]
\]

\[= \left( \frac{1}{v_1 \cdot (v_2 \times v_3)} \right)^2 [((v_1 \cdot (v_2 \times v_3)) v_1]
\]

\[= \frac{1}{v_1 \cdot (v_2 \times v_3)} v_1.\]
Then

\[
\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \mathbf{k}_1 \cdot \left( \frac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \mathbf{v}_1 \right)
\]

\[
= \left( \frac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \right) (\mathbf{k}_1 \cdot \mathbf{v}_1)
\]

\[
= \frac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}.
\]