Show that the line integral
\[ \int_C (1 - ye^{-x}) \, dx + e^{-x}dy \]
where \( C \) is any path from \((0, 1)\) to \((1, 2)\) is independent of path and evaluate the integral.

**Solution**

Let \( f(x, y) = x + ye^{-x} \). Then, it is easy to check that \( \mathbf{F} = \nabla f \).

Note that a path is a piecewise-smooth curve. So, we can write \( C = C_1 \cup C_2 \cup \cdots \cup C_n \),
where \( C_i \) is a smooth curve for all \( i = 1, 2, \cdots, n \). Let \( a_i \) be the initial point of \( C_i \)
and \( b_i \) be the end point of \( C_i \). Since \( C \) is continuous, we know that \( b_{i-1} = a_i \) for all \( i = 2, 3, \cdots, n \). Also, for all \( i \), since \( C_i \) is smooth and \( f \) is of \( C^1 \), by Fundamental Theorem of Line Integrals, we have

\[ \int_{C_i} \mathbf{F} \cdot d\mathbf{r} = f(b_i) - f(a_i). \]

Thus,

\[ \int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \cdots + \int_{C_n} \mathbf{F} \cdot d\mathbf{r} \]

\[ = [f(b_1) - f(a_1)] + [f(b_2) - f(a_2)] + \cdots + [f(b_n) - f(a_n)] \]

\[ = f(b_n) - f(a_1). \]

Note that since \( a_1 \) is the initial point of \( C_1 \), it is the initial point of \( C \), which is \((0, 1)\) and since \( b_n \) is the end point of \( C_n \), it is the end point of \( C \), which is \((1, 2)\). So,

\[ \int_{C} \mathbf{F} \cdot d\mathbf{r} = f(b_n) - f(a_1) = f(1, 2) - f(0, 1) = \left[ (1) + (2)e^{-1} \right] - \left[ (0) + (1)e^{-0} \right] = \frac{2}{e}, \]

which is a constant no matter which path \( C \) is. Therefore, line integral is independent of path.

**13.3 #22** Find the work done by the force field \( \mathbf{F}(x, y) = e^{-y}i - xe^{-y}j \) in moving an object from \( P(0, 1) \) to \( Q(2, 0) \).

**Solution**

Let \( f(x, y) = xe^{-y} \). Then, it is easy to check that \( \mathbf{F} = \nabla f \). The work done is

\[ W = \int_{C} \mathbf{F} \cdot d\mathbf{s} = \int_{C} \mathbf{F} \cdot d\mathbf{r} \]

Since \( f \) is of \( C^1 \) and \( \mathbf{F} = \nabla f \), this line integral is independent of path. Thus, the work done is

\[ W = \int_{C} \nabla f \cdot d\mathbf{r} = f(2, 0) - f(0, 1) = (2)e^{-0} - (0)e^{-1} = 2. \]
13.3 #26 Let \( \mathbf{F} = \nabla f \), where \( f(x, y) = \sin(x - 2y) \). Find curves \( C_1 \) and \( C_2 \) that are not closed and satisfy the equation
(a) \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0 \).
(b) \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1 \).

[Solution]
(a) If \( C_1 \) is a smooth curve from \( (0, 0) \) to \( (2\pi, 0) \), then \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = f(2\pi, 0) - f(0, 0) = 0 \). So, choose \( C_1 \) to be the line segment from \( (0, 0) \) to \( (2\pi, 0) \). It is obviously that \( C_1 \) is not closed.
(b) If \( C_2 \) is a smooth curve from \( (0, 0) \) to \( \left( \frac{\pi}{2}, 0 \right) \), then \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f \left( \frac{\pi}{2}, 0 \right) - f(0, 0) = 1 \). So, choose \( C_1 \) to be the line segment from \( (0, 0) \) to \( \left( \frac{\pi}{2}, 0 \right) \). It is obviously that \( C_2 \) is not closed.

13.3 #30 Determine whether or not the set \( \{(x, y) \mid x \neq 0\} \) is
(a) open.
(b) connected.
(c) simply-connected.

[Solution]
(a) It is open since there is no boundary in this set.
(b) It is not connected since the line \( x = 0 \) separates the set into two pieces.
(c) Since it is not connected, it is not simply-connected.

13.4 #2 Evaluate the line integral
\[
\oint_C y \, dx - x \, dy
\]
where \( C \) is the circle with center the origin and radius 1 by two methods: directly, and using Green’s Theorem.

[Solution]
To calculate it directly, we parametrize \( C \) as \( x = \cos \theta \) and \( y = \sin \theta \) where \( 0 \leq \theta \leq 2\pi \). Thus,
\[
\oint_C y \, dx - x \, dy = \int_0^{2\pi} \left( \sin \theta \right) \left( -\sin \theta \, d\theta \right) - \left( \cos \theta \right) \left( \cos \theta \, d\theta \right) = \int_0^{2\pi} (-1) \, d\theta = -2\pi.
\]
The Green’s theorem tells us that
\[
\oint_C y \, dx - x \, dy = \iint_D \left( \frac{\partial (-x)}{\partial x} - \frac{\partial (y)}{\partial y} \right) \, dA = \int_D -2 \, dA = -2A(D)
\]
where \( D \) is the region enclosed by \( C \) which is a disk. Therefore, the area of \( D \) is \( A(D) = \pi (1)^2 = \pi \). So,
\[
\oint_C y \, dx - x \, dy = -2A(D) = -2\pi.
\]
13.4 #8 Use Green’s Theorem to evaluate the line integral $\int_C x^2y^2 \, dx + 4xy^3 \, dy$ along the given positively oriented curve $C$ which is the triangle with vertices (0,0), (1,3), and (0,3).

**[Solution]**

Let $D$ be the region enclosed by the triangle $C$. It can be described as $0 \leq x \leq 1$ and $3x \leq y \leq 3$. By the Green’s theorem, we have

$$\int_C x^2y^2 \, dx + 4xy^3 \, dy = \iint_D \left( \frac{\partial}{\partial x} (4xy^3) - \frac{\partial}{\partial y} (x^2y^2) \right) \, dA = \iint_D (4y^3 - 2x^2y) \, dA$$

$$= \int_0^1 \int_{3x}^3 (4y^3 - 2x^2y) \, dy \, dx = \frac{318}{5}.$$  

13.4 #10 Use Green’s Theorem to evaluate the line integral $\int_C xe^{-2x} \, dx + (x^4 + 2x^2y^2) \, dy$ along the given positively oriented curve $C$ which is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

**[Solution]**

Let $C_1$ be the circle $x^2 + y^2 = 1$ and $C_2$ be the circle $x^2 + y^2 = 4$. So, $C = C_1 \cup C_2$. Also, let $D$ be the region in between. By the discussion in page 937 for the Green’s Theorem, we know that

$$\iiint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_{C_1} P \, dx + Q \, dy + \int_{C_2} P \, dx + Q \, dy = \int_C P \, dx + Q \, dy.$$  

Therefore,

$$\int_C xe^{-2x} \, dx + (x^4 + 2x^2y^2) \, dy = \iint_D \left( \frac{\partial (x^4 + 2x^2y^2)}{\partial x} - \frac{\partial (xe^{-2x})}{\partial y} \right) \, dA = \iint_D (4x^3 + 4xy^2) \, dA.$$  

Now, $D$ can be described as $x = r \cos \theta$ and $y = r \sin \theta$, where $1 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. Hence,

$$\int_C xe^{-2x} \, dx + (x^4 + 2x^2y^2) \, dy = \iint_D 4x (x^2 + y^2) \, dA = \int_0^{2\pi} \int_1^2 (4r \cos \theta) (r^2) \, r \, dr \, d\theta$$

$$= \int_1^2 \int_0^{2\pi} 4r^4 \cos \theta \, d\theta \, dr = 0.$$  

13.4 #14 Use Green’s Theorem to evaluate $\int_C \mathbf{F} \cdot \, d\mathbf{r}$ where $\mathbf{F} (x,y) = (y^2 \cos x, x^2 + 2y \sin x)$ and $C$ is the triangle from (0,0) to (2,6) to (2,0) to (0,0). (Check the orientation of the curve before applying the theorem.)

**[Solution]**

Let $D$ be the region enclosed by the triangle $C$. It can be described as $0 \leq x \leq 2$ and $0 \leq y \leq 3x$. Note that $C$ is travelling clockwisely. But, $C$ is a piecewise-smooth, simple
closed curve. So, to apply the Green’s Theorem, we need $-C$. Thus,

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left( \frac{\partial}{\partial x} (x^2 + 2y \sin x) - \frac{\partial}{\partial y} (y^2 \cos x) \right) dA = \iint_D \left( (2x + 2y \cos x) - (2y \cos x) \right) dA$$

$$= \iint_D 2xdA = \int_0^2 \int_0^{3x} 3xdydx = 24.$$

Therefore,

$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -24.$$

13.4 #18 A particle starts at the point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and then along the semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green’s Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = (x, x^3 + 3xy^2)$.

**[Solution]**

Let $D$ be the region enclosed by the path of the particle. It is an upper-half disk with radius 2. Also, the particle travels counterclockwise. Since the path is a positively oriented, piecewise-smooth, simple closed curve, the Green’s Theorem applies. Thus, the work done is

$$W = \int_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left( \frac{\partial}{\partial x} (x^3 + 3xy^2) - \frac{\partial}{\partial y} x \right) dA = \iint_D \left( 3x^2 + 3y^2 \right) dA.$$

Since $D$ is an upper-half disk with radius 2, we can parametrize it as $x = r \cos \theta$ and $y = r \sin \theta$, where $0 \leq r \leq 2$ and $0 \leq \theta \leq \pi$. So, we have

$$W = \iint_D (3x^2 + 3y^2) dA = \int_0^2 \int_0^\pi (3r^2) r \, d\theta \, dr = \int_0^2 \int_0^\pi 3r^3 \, d\theta \, dr = 12\pi.$$

13.4 #28 Complete the proof of the special case of Green’s Theorem by proving Equation 3.

**[Solution]**