1.1

19. Consider the linear system

\[
\begin{align*}
    x + y - z &= -2 \\
    3x - 5y + 13z &= 18 \\
    x - 2y + 5z &= k
\end{align*}
\]

where \( k \) is an arbitrary number.

(a) For which values of \( k \) does the system have one or infinitely many solutions?

To answer this, we turn the system into an augmented matrix and put it in reduced row echelon form:

\[
\begin{pmatrix}
    1 & 1 & -1 & -2 \\
    3 & -5 & 13 & 18 \\
    1 & -2 & 5 & k
\end{pmatrix}
\]

\( \rightarrow \)

\[
\begin{pmatrix}
    1 & 1 & -1 & -2 \\
    0 & -8 & 14 & 24 \\
    0 & -3 & 6 & k+2
\end{pmatrix}
\]

\( \rightarrow \)

\[
\begin{pmatrix}
    1 & 0 & 1 & 1 \\
    0 & 1 & -2 & -3 \\
    0 & 0 & 0 & k-7
\end{pmatrix}
\]

The matrix is now in rref. There are no solutions if \( k-7 \neq 0 \). Thus, there will be at least one solution if and only if \( k=7 \).

(b) If \( k=7 \), how many solutions does the system have?

If \( k=7 \), the matrix looks like

\[
\begin{pmatrix}
    1 & 0 & 1 & 1 \\
    0 & 1 & -2 & -3 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]

There are two pivot variables and one free variable, so there are infinitely many solutions when \( k=7 \).
(c) Find all solutions when \( k = 7 \).

The second row gives the equation \( y - 2z = -3 \).

The first row gives the equation \( x + z = 1 \).

Setting \( z = t \) to be our free variable,

we have \( x = 1 - t \) and \( y = 2 + 3t \).

so \( \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - t \\ 2 + 3t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}t \) gives all solutions as \( t \) varies.

20. For which values of \( a, b, c, d, \) and \( e \) is the following matrix in reduced row-echelon form?

\[
A = \begin{pmatrix}
0 & 0 & 2 & 1 & b \\
0 & 0 & 0 & c & d \\
0 & 0 & e & 0 & 0
\end{pmatrix}
\]

\( e \) must be zero since the row above it has only zeros to the left. \( a \) must be 1 since otherwise the first row would not have a leading 1. \( c \) must be zero since if it were a leading 1, the 1 above it would have to be zero. \( d \) can be 0 or 1.

If \( d = 0 \), \( b \) can be anything.

If \( d = 1 \), it is a leading 1, so \( b \) must be zero.

So the two possibilities are \((a, b, c, d, e) = (1, b, 0, 0, 0)\) for any \( b \)

and \((1, 0, 0, 1, 0)\).
36. Find all vectors in \( \mathbb{R}^3 \) perpendicular to \( \left( \begin{array}{c} 1 \\ 3 \\ -1 \end{array} \right) \). Draw a sketch.

If a vector \( \left( \begin{array}{c} x \\ y \\ z \end{array} \right) \) is perpendicular to \( \left( \begin{array}{c} 1 \\ 3 \\ -1 \end{array} \right) \), then it satisfies the equation \( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = 0 \)

That is, \( x + 3y - z = 0 \)

or, \( x + 3y - z = 0 \) must be true.

Then \( x = -3y + z \), and so \( x \) is completely determined by \( y \) and \( z \) (our free variables).

If we set \( y = s \) and \( z = t \), we have

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3s + t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} t.
\]

Thus, all vectors perpendicular to \( \left( \begin{array}{c} 1 \\ 3 \\ -1 \end{array} \right) \) have this form. Geometrically, we can think of the collection

\[
\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} t \left| \begin{array}{c} s, t \in \mathbb{R} \end{array} \right. \right\}
\]

as a plane spanned by the two vectors \( \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \).
6. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be the vectors in $\mathbb{R}^2$ drawn below.

How many solutions does the system $x \vec{v}_1 + y \vec{v}_2 = \vec{v}_3$ have?

Geometrically, any multiple $x \vec{v}_1$ will point in the same direction as $\vec{v}_1$, and similarly for any multiple $y \vec{v}_2$.

Since $\vec{v}_1$ and $\vec{v}_2$ point the same direction, the sum $x \vec{v}_1 + y \vec{v}_2$ will still point the same direction as $\vec{v}_1$ and $\vec{v}_2$:

In particular, $x \vec{v}_1 + y \vec{v}_2$ cannot point the same direction as $\vec{v}_3$, so we cannot have $x \vec{v}_1 + y \vec{v}_2 = \vec{v}_3$ for any choice of $x$ and $y$. So there are no solutions.
25. Let $A$ be a $4 \times 4$ matrix and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. We are told that the system $A\vec{x} = \vec{b}$ is inconsistent. What can we say about the number of solutions of the system $A\vec{x} = \vec{c}$?

Since the system is inconsistent, the $\text{rref}$ of $A$ must have a row of zeros

$$\text{rref}(A) = \begin{pmatrix} 
\text{---} \\
0 & 0 & 0 & 0 
\end{pmatrix}$$

and if $\vec{b} = \begin{pmatrix} b_1 \\
b_2 \\
b_3 \\
b_4 
\end{pmatrix}$

becomes

$$\vec{b}' = \begin{pmatrix} b_1' \\
b_2' \\
b_3' \\
b_4' 
\end{pmatrix}$$

after row reduction.

Then the augmented matrix must look like

$$\begin{pmatrix} 
\text{---} \\
| b_1' \\
\text{---} \\
0 & 0 & 0 & 0 \end{pmatrix}$$

with $b_4' \neq 0$ since the system is inconsistent.

Now let's replace $\vec{b}$ with $\vec{c} = \begin{pmatrix} c_1 \\
c_2 \\
c_3 \\
c_4 
\end{pmatrix}$ in the above and say $\vec{c}$ becomes

$$\vec{c}' = \begin{pmatrix} c_1' \\
c_2' \\
c_3' \\
c_4' 
\end{pmatrix}$$

after row reduction.
(where \( \text{rref}(A) \) still stays the same).

$$
\begin{bmatrix}
\vdots \\
\vdots \\
0 & 0 & 0 & 0 & c_4' \\
\end{bmatrix}
$$

If \( c_4' \neq 0 \), the system \( A \mathbf{x} = \mathbf{c} \) will remain inconsistent. If \( c_4' = 0 \), the bottom row will yield the uninformative equation \( 0 = 0 \). So we will really have three equations in four variables. Thus, a unique solution is impossible. So \( A \mathbf{x} = \mathbf{c} \) will have either infinitely many or no solutions.