5. Multiply, if possible:
\[
\begin{bmatrix}
0 & b \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} =
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]
\[3 \times 2 \times 2 = 2 \times 2\]

8. Multiply from front:
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix} =
\begin{bmatrix}
ad-bc & -ac \\
cd-bc & -ad+bc
\end{bmatrix}
\]

13. Multiply from front:
\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & k
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix} =
\begin{bmatrix}
a & b & c \\
g & h & k \\
d & e & f
\end{bmatrix}
\]

18. * graded — see solutions to graded problems at the end

29. * graded ...

46. Find a 2x2 matrix \( A \) such that \( A^2 = A \) and all entries of \( A \) are non-zero.

Solution: Let’s translate this into geometry. We want a transformation which, when applied to a vector twice, gives the same result as when it is applied once. The example that stands out is projection onto a line. When we project once, every vector ends up on the line. When we project again, every vector is already on the line, so it stays the same.

Let’s find a matrix representing projecting onto the line \( y = x \) in the direction of the vector \( \mathbf{w} = (1) \) (there are many choices here — as long as you don’t choose the \( x \) or \( y \) axis, the matrix will end up with all nonzero entries).

A unit vector in the direction of our line is \( \mathbf{e} = \left( \frac{1}{\sqrt{2}} \right) \)

To find the matrix representing projection onto this line, we need to find the projections of \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) onto \( \mathbf{e} \).
We have \( \text{proj}_V (\hat{e}_1) = (\hat{e}_1 \cdot \hat{u}) \hat{u} = \left( \frac{1}{2} \cdot \left( \frac{1}{\sqrt{3}} \right) \right) \hat{u} = \frac{1}{\sqrt{2}} \hat{u} = \left( \frac{\sqrt{2}}{2} \right) \)

and \( \text{proj}_V (\hat{e}_2) = (\hat{e}_2 \cdot \hat{u}) \hat{u} = \left( \frac{1}{2} \cdot \left( \frac{1}{\sqrt{3}} \right) \right) \hat{u} = \frac{1}{\sqrt{2}} \hat{u} = \left( \frac{\sqrt{2}}{2} \right) \)

Thus, \( A = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \). Let’s check that \( A^2 = A \):

\[
A^2 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + \frac{1}{4} & \frac{1}{4} + \frac{1}{4} \\ \frac{1}{4} + \frac{1}{4} & \frac{1}{4} + \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = A
\]

5.6. Find all matrices \( X \) that satisfy

\[
X \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 2 \end{pmatrix}_{2 \times 2}
\]

First, notice that the dimensions of \( X \) must be \( 2 \times 2 \).

We write \( X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and determine what \( a, b, c, d \) must be. We have

\[
X \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} a + 3b & 2a + 5b \\ c + 3d & 2c + 5d \end{pmatrix}
\]

and \( I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Thus, we must have the following system of equations:

\[
\begin{align*}
1 & : a + 3b = 1 \\
2 & : 2a + 5b = 0 \\
3 & : c + 3d = 0 \\
4 & : 2c + 5d = 1
\end{align*}
\]

We can solve this system by hand pretty quickly:

\[ 2a + 5b = 0 \Rightarrow a = -\frac{5}{2}b \]

Thus, plugging into (2) we have
\[-\frac{5}{2} b + 3b = 1\]

or \[-\frac{5}{2} b + \frac{6}{2} b = \frac{1}{2} = 1, \text{ so } b = 2\]

Then \[a = -\frac{5}{2} - 2 = -\frac{9}{2}\]

Similarly, from (3) we have \[c = -3d\]. Plugging into (4) we get \[2(-3d) + 5d = 1\]

so \[-d = 1\]. Thus, \[d = -1\] and \[c = -3(-1) = 3\]

Thus, \[X\] must be

\[
X = \begin{pmatrix}
-5 & 2 \\
3 & -1
\end{pmatrix}.
\]

\[2.4\]

4.8 Determine whether the following matrices are invertible and find their inverses:

4. \[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}.
\]

We form the augmented matrix and row-reduce:

\[
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 2 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 2 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Thus, the matrix is invertible and the inverse is

\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{bmatrix}.
\]

8. \[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 3 & 6
\end{bmatrix}
\]

We form the augmented matrix and row-reduce:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 2 & 3 & 0 & 1 & 0 \\
1 & 3 & 6 & 0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & -1 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & -1 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & -1 & 1 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & -1 & 2 & -1 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 2 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 3 & -3 & 1 \\
0 & 1 & 0 & -3 & 5 & -2 \\
0 & 0 & 1 & -2 & 1
\end{bmatrix}.
\]
Thus, the matrix is invertible and its inverse is given by
\[
\begin{bmatrix}
3 & -3 & 1 \\
-3 & 5 & -2 \\
1 & -2 & 1
\end{bmatrix}.
\]

29. For which values of \( k \) is
\[
\begin{bmatrix}
1 & 0 & k \\
1 & 2 & k^2 \\
2 & 4 & k^2
\end{bmatrix}
\]
invertible?

Let's augment and row-reduce to find the answer:

\[
\begin{bmatrix}
1 & 0 & k \\
1 & 2 & k^2 \\
1 & 4 & k^2
\end{bmatrix}
\xrightarrow{-1\times Row_1}
\begin{bmatrix}
1 & 0 & k \\
0 & 2 & k^2 \\
0 & 0 & k^2-k
\end{bmatrix}
\xrightarrow{-\frac{1}{2}\times Row_2}
\begin{bmatrix}
1 & 0 & k \\
0 & 1 & \frac{k^2}{2} \\
0 & 0 & 1
\end{bmatrix}
\xrightarrow{k-\frac{k^2}{2}+\frac{k^2}{2}}
\begin{bmatrix}
1 & 0 & k \\
0 & 1 & \frac{k^2}{2} \\
0 & 0 & 1
\end{bmatrix}
\]

The next step would be to divide row 3 by \( k^2-3k+2 \),
but first we have to think about what happens when \( k^2-3k+2 = 0 \).
\( k^2-3k+2 = (k-2)(k-1) \) so it is zero when \( k=1 \) or \( k=2 \)
in which case the bottom row contains only zeros so the matrix cannot be reduced further and is thus not invertible.

If \( k \neq 1, 2 \), we may divide by \( k^2-3k+2 = (k-2)(k-1) \).

\[
\begin{bmatrix}
1 & 0 & 2-k \\
0 & 1 & k-1 \\
0 & 0 & 1
\end{bmatrix}
\xrightarrow{\frac{3}{2-k}}
\begin{bmatrix}
1 & 0 & 2-k \\
0 & 1 & k-1 \\
0 & 0 & 1
\end{bmatrix}
\]

Thus, the matrix is invertible for all \( k \) besides \( k=1, 2 \)
and the inverse is given by
35. Consider

\[
A = \begin{bmatrix}
a & b & c \\
0 & d & e \\
0 & 0 & f \\
\end{bmatrix}
\]

a) For which values of \(a, b, c, d, e, \) and \(f\) is \(A\) invertible?

To start, \(a\) must be nonzero since otherwise the matrix cannot be fully row-reduced (it will not have rank 3).

Moving to the next column, \(d\) is the only candidate to become a leading 1, so it must be nonzero also.

Finally, \(f \neq 0\) for the same reason.

Thus, \(A\) is invertible if and only if \(a, d, \) and \(f\) are all nonzero.

b) More generally, an \(n \times n\) upper triangular matrix \(A\) is invertible if and only if:

\[\begin{array}{c}
\text{it has rank } n \\
\text{rref}(A) \text{ has } n \text{ leading 1's} \\
\text{all diagonal entries of } A \text{ are nonzero.}
\end{array}\]
Let's show that if a square matrix $A$ has two equal columns, then $A$ is not invertible.

Suppose $A$ is a square $n \times n$ matrix with two equal columns (say the $i$th and $j$th columns).

To get some intuition, let's show $A$ is not invertible if $n = 2$.

Then $A = \begin{bmatrix} a & a \\ b & b \end{bmatrix}$ (suppose $a \neq 0$ or we're done)

We reduce:

$\begin{bmatrix} a & a \\ b & b \end{bmatrix} - \frac{b}{a} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$

So $A$ does not have full rank, so it cannot be invertible!

Now let's work this out for general $n$.

When we attempt to row-reduce $A$, everything that happens to the $i$th column will also happen to the $j$th column, so they will remain equal throughout. Thus $\text{rref}(A)$ will have two equal columns. But $A$ is invertible if and only if $\text{rref}(A) = I_n$, and $I_n$ does not have any two columns equal! Thus $A$ cannot be invertible.

\[ \square \]

41. Graded ...
S 6. Let \( A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \)

Find \( A^{-1} \) and interpret \( A \) and \( A^{-1} \) geometrically.

For any matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), its inverse is given by

\[
\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

In this case, we see that

\[
A^{-1} = \frac{1}{\cos^2 \alpha + \sin^2 \alpha} \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}
\]

(A is always invertible since \( ad-bc = \cos^2 \alpha + \sin^2 \alpha = 1 \))

We already know how to interpret \( T(\mathbb{R}^2) = A \mathbb{R}^2 \) geometrically: \( A \) is the matrix that represents rotation by an angle of \( \alpha \). Thus, \( A^{-1} \), which should undo what \( A \) does, should represent rotation back to the original spot, i.e. rotation by an angle of \(-\alpha\). Indeed, if we write

\[
A^{-1} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix}
\]

(using the fact that \( \sin \) is an odd function and \( \cos \) is an even function)

we see that \( A^{-1} \) does in fact represent rotation by an angle of \(-\alpha\). \( \Box \)
6. \[ A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \].

To see what \( A \) represents geometrically, let's see where \( A \) sends \((1, 0)\) and \((0, 1)\).

\[ A(1, 0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad A(0, 1) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

We see that \( A \) rotates both vectors by an angle of \(-\frac{\pi}{4}\) and scales them by a factor of \( \sqrt{2} \).

In fact, we may write \( A \) as

\[
\sqrt{2} \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix} \quad \text{where} \quad \omega = -\frac{\pi}{4}
\]

\[
= \sqrt{2} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
\]

We expect \( A^{-1} \) to undo this by scaling by a factor of \( \frac{1}{\sqrt{2}} \) and rotating by an angle of \( \frac{\pi}{4} \). That is, we expect

\[
A^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \quad \text{where} \quad \beta = \frac{\pi}{4}
\]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}
\]
Indeed, if we use the formula $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

we see that in this case

$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ turns out to be exactly what we expected. \(\blacksquare\)

68. * graded
18. Find all matrices that commute with the given matrix $A$

$$A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

Solution: Let $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ be a matrix that commutes with $A$.

We have

$$\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2b_{11}+3b_{21} & 2b_{12}+3b_{22} \\ -3b_{11}+2b_{21} & -3b_{12}+2b_{22} \end{pmatrix} = \begin{pmatrix} 2b_{11}-3b_{21} & 3b_{11}+2b_{12} \\ 2b_{21}-3b_{22} & 3b_{21}+2b_{22} \end{pmatrix}$$

Therefore we have:

$$\begin{cases} 2b_{11}+3b_{21} = 2b_{11}-3b_{21} \\ 2b_{12}+3b_{22} = 3b_{11}+2b_{12} \\ -3b_{11}+2b_{21} = 2b_{21}-3b_{22} \\ -3b_{12}+2b_{22} = 3b_{21}+2b_{22} \end{cases}$$
Simplify the system above, we get:
\[ \begin{align*}
& b_{11} + b_{12} = 0 \\
& b_{11} - b_{22} = 0
\end{align*} \]
so we can set: \( b_{11} = t, \quad b_{21} = s, \)
then \( b_{22} = t, \quad b_{12} = -s. \)
The matrices that commutes with \( A \) having the form \( \begin{pmatrix} t & -s \\ s & t \end{pmatrix} \) where \( t, s \in \mathbb{R}. \)

29. Consider the matrix \( D_\alpha = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \). We know that the linear transformation \( T(x) = D_\alpha x \) is a counterclockwise rotation through an angle \( \alpha. \)

a. For two angles, \( \alpha \) and \( \beta, \) consider the products \( D_\alpha D_\beta \) and \( D_\beta D_\alpha. \) Arguing geometrically, describe the linear transformations \( \vec{y} = D_\alpha D_\beta \vec{x} \) and \( \vec{y} = D_\beta D_\alpha \vec{x}. \) Are the two transformations the same?

b. Now compute the products \( D_\alpha D_\beta \) and \( D_\beta D_\alpha. \) Do the results make sense in terms of your answer in part (a)?
Recall the trigonometric identities:
\[
\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.
\]
Solution: (a). They are the same. \( \tilde{y} = D_\alpha D_\beta \tilde{x} \) means you first rotate \( \tilde{x} \) by \( \beta \), then rotate it by \( \alpha \). \( \tilde{y} = D_\beta D_\alpha \tilde{x} \) means you first rotate \( \tilde{x} \) by \( \alpha \), then rotate it by \( \beta \). So the effect of both transformations is to rotate \( \tilde{x} \) by \( \alpha + \beta \).

(b) \[ D_\alpha D_\beta = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \]

\[ = \begin{bmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -\cos \alpha \sin \beta - \sin \alpha \cos \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \end{bmatrix} \]

\[ = \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix} \]

\[ D_\beta D_\alpha = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \]

\[ = \begin{bmatrix} \cos \beta \cos \alpha - \sin \beta \sin \alpha & -\cos \beta \sin \alpha - \sin \beta \cos \alpha \\ \sin \beta \cos \alpha + \cos \beta \sin \alpha & -\sin \beta \sin \alpha + \cos \beta \cos \alpha \end{bmatrix} \]

\[ = \begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix} \]
We see $Dx \cdot D\theta = D\theta \cdot Dx$, so our answer in a is correct.

2.4

19. Decide whether the linear transformation is invertible. Find the inverse transformation if it exists. Do the computations with paper and pencil. Show all your work.

$Y_1 = X_1 + X_2 + X_3$
$Y_2 = X_1 + 2X_2 + 3X_3$
$Y_3 = X_1 + 4X_2 + 9X_3$

Solution: \[
\begin{pmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 4 & 9
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
X_3
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 2 & 3 & 0 & 1 \\
1 & 4 & 9 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 2 & -1 & 10 \\
0 & 3 & 8 & -1 & 0 & 1
\end{pmatrix}
\]
\[ \begin{align*}
\frac{0-3}{0-3x^2} & \rightarrow \begin{pmatrix}
1 & 0 & 1 & 2 & -1 & 0 \\
0 & 1 & 2 & -1 & 1 & 0 \\
0 & 0 & 1 & 1 & -\frac{3}{2} & \frac{1}{2}
\end{pmatrix} \\
\frac{3x^\frac{1}{2}}{} & \rightarrow \begin{pmatrix}
1 & 0 & 1 & 2 & -1 & 0 \\
0 & 1 & 2 & -1 & 1 & 0 \\
0 & 0 & 1 & 1 & -\frac{3}{2} & \frac{1}{2}
\end{pmatrix} \\
\frac{0+3}{0-3x^2} & \rightarrow \begin{pmatrix}
1 & 0 & 0 & 3 & -\frac{5}{2} & \frac{1}{2} \\
0 & 1 & 0 & -3 & 4 & -1 \\
0 & 0 & 1 & 1 & -\frac{3}{2} & \frac{1}{2}
\end{pmatrix}
\end{align*} \]

Therefore, it is invertible. Its inverse transformation is

\[ \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
3 & -\frac{5}{2} & \frac{1}{2} \\
-3 & 4 & -1 \\
1 & -\frac{3}{2} & \frac{1}{2}
\end{pmatrix} \begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} \]
41. Which of the following linear transformations $T$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ are invertible? Find the inverse if it exists.

a. Reflection about a plane.
b. Orthogonal projection onto a plane.
c. Scaling by a factor 5. [i.e., $T(\mathbf{v}) = 5\mathbf{v}$ for all vectors $\mathbf{v}$].
d. Rotation about an axis.

**Solution:**

a. It is invertible. After reflecting a vector, if we reflect it again, we will get the original vector back. So the inverse of a reflection is the reflection itself.

b. It is not invertible. For any two different vectors which are perpendicular to the plane, if we project them onto the plane, we will get $0$. 
so orthogonal projection is not one-to-one. So it is not invertible.

C. It is invertible. Obviously scaling by a factor $\frac{1}{5}$ is its inverse.

d. It is invertible. After rotating a vector, we can always get it back by rotating about the same axis in opposite direction by the same angle, which is the inverse transformation.

68. For two invertible, nxn matrices $A$ and $B$, determine if the formula stated below is necessarily true.

$$(A-B)(A+B) = A^2 - B^2$$

Solution: it is not necessarily true.

$$(A-B)(A+B) = A^2 - BA + AB - B^2$$

So if $AB \neq BA$, the statement
above is false. A counterexample is as follows. \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) \( B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \)

\( AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) \( BA = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \)

Both \( A \) and \( B \) are invertible, but \( AB \neq BA \).