This exam contains 11 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- **Follow the instructions closely**. For example, if you are asked to justify your answers, then do so in a brief and coherent way.

- **Points will be taken off for incorrect statements, even if correct ones are present**. Be careful about what you include in your answers. If they contain both the correct answers and incorrect or nonsense statements, points will be taken off.

- If you need more space, use the back of the pages; clearly indicate when you have done this.

**Good luck!!** Do not write in the table to the right.
1. (20 points) **Solving linear systems**: show your work.

(a) (10 points) Find all solutions to the linear system

\[
\begin{align*}
  x + 4y + z &= 6 \\
  x + y &= 2 \\
  x - y - z &= -1
\end{align*}
\]

**Solution:** We express the system as an augmented matrix and row reduce:

\[
\begin{pmatrix}
  1 & 4 & 1 & 6 \\
  1 & 1 & 0 & 2 \\
  1 & -1 & -1 & -1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  1 & 4 & 1 & 6 \\
  0 & -3 & -1 & -4 \\
  0 & -5 & -2 & -7
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  1 & 4 & 1 & 6 \\
  0 & 1 & 1/3 & 4/3 \\
  0 & 0 & 1/3 & 1/3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  1 & 0 & 0 & 1 \\
  0 & 1 & 0 & 1 \\
  0 & 0 & 1 & 1
\end{pmatrix}
\]

Thus \( x = y = z = 1 \) is the only solution.
(b) (10 points) Consider the linear system

\[
\begin{align*}
x + y - z &= 0 \\
3x - 5y + 13z &= 0
\end{align*}
\]

Does this system have a unique solution, infinitely many solutions, or no solutions at all? Justify your answer. (You don’t have to solve the equation.)

**Solution:** Because there are more variables than equations we cannot have exactly one solution. Since \(x = y = z = 0\) is a solution, we cannot have zero solutions either. Hence there must be infinitely many solutions. Alternatively we can solve the system of equations to get \(x = -t, y = 2t,\) and \(z = t,\) which gives an infinite family of solutions.
2. (20 points) Let $A$ be a $3 \times 3$ matrix, and let $\vec{b}$ be a vector in $\mathbb{R}^3$ such that $A\vec{x} = \vec{b}$ is inconsistent (admits no solutions).

(a) (5 points) What are the possible values of the rank of $A$? (Recall that the rank is the number of pivot variable)

**Solution:** The rank of $A$ can not be 3, otherwise the system must admit a unique solution. The rank of $A$ can be 0, 1 or 2. For example, let $\vec{b} = (1, 1, 1)^T$, then $A$ can be $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. For each of the matrix above, the system is inconsistent.

Grading: Giving 0, 1, 2 as possible values will get full credit. If 0 is omitted, it will get 4 points. Other solutions get 2 points at most.

(b) (5 points) For $A$ as above, how many solutions are there to $A\vec{x} = \vec{0}$?

**Solution:** Since $A$ is not full rank, if right hand side is 0, the system has infinitely many solutions.

Grading: Giving "infinitely many solutions" will get full credit. Otherwise at most 2 or 3 points.
(c) (10 points) Express the image of the matrix
\[
\begin{pmatrix}
1 & -1 \\
2 & -1 \\
2 & 1
\end{pmatrix}
\]
as the kernel of a matrix $B$.

**Solution:** We would follow the hint in Exercise 3.1 problem 42. We write the system as
\[
\begin{align*}
x_1 - x_2 &= y_1 \\
2x_1 - x_2 &= y_2 \\
2x_1 + x_2 &= y_3.
\end{align*}
\]
Then using the Gaussian elimination, we can simplify the system as
\[
\begin{align*}
x_1 &= y_2 - y_1 \\
x_2 &= y_2 - 2y_1 \\
0 &= 4y_1 - 3y_2 + y_3.
\end{align*}
\]
Therefore the image $(y_1, y_2, y_3)^T$ satisfy the equation $4y_1 - 3y_2 + y_3 = 0$. So the image is the kernel of matrix $\begin{pmatrix} 4 & -3 & 1 \end{pmatrix}$.

Grading: Giving the numbers $4, -3, 1$ will get at least 8 points. Just giving the RREF
\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\]
specifying the image is the span of column vectors will get 4-5 points. Otherwise it will get at most 2 points.
3. (20 points) **Linear transformations of the plane**

(a) (5 points) Write down the $2 \times 2$ matrix of the linear transformation which gives projection onto the diagonal line $x = y$.

**Solution:** We look at the unit vector lying on the line $x = y$, \[
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix}.
\]

Given a unit vector \[
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]
lying on a line, the matrix of the transformation projecting onto the line spanned by the unit vector is \[
\begin{bmatrix}
u_1^2 & u_1 u_2 \\
u_1 u_2 & u_2^2
\end{bmatrix}
\]

Plugging our values for $u_1$ and $u_2$ into the matrix, we get

\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]

Note: using the unit vector along the line in another direction will give the same matrix. Using a non-unit vector such as \[
\begin{bmatrix}1 \\1
\end{bmatrix}
\]
will result in a projection matrix multiplied by a scaling matrix, and in particular it will not project points on the line to themselves.

(b) (5 points) Write down the $2 \times 2$ matrix $A$ of the linear transformation which gives reflection over the diagonal line $x = y$.

**Solution:** Given a line $\ell$, in geometric terms the transformation giving reflection of a vector $\vec{x}$ over $\ell$ is $\vec{x}^\parallel - \vec{x}^\perp$, or equivalently $2\vec{x}^\parallel - \vec{x}$. Using the fact from part (a) that

\[
\vec{x}^\parallel = \begin{bmatrix}1/2 & 1/2 \\
1/2 & 1/2
\end{bmatrix} \vec{x},
\]

we can conclude that

\[
\text{ref}_\ell(\vec{x}) = 2\vec{x}^\parallel - \vec{x}
\]

\[
= 2 \cdot \begin{bmatrix}1/2 & 1/2 \\
1/2 & 1/2
\end{bmatrix} \vec{x} - \begin{bmatrix}1 & 0 \\
0 & 1
\end{bmatrix} \vec{x}
\]

\[
= \begin{bmatrix}0 & 1 \\
1 & 0
\end{bmatrix} \vec{x}
\]

This matrix gives the desired transformation.

Alternatively, use the fact that $T(e_1) = e_2$ and $T(e_2) = e_1$, which gives an equivalent matrix.

Note: Even though the matrix is of the form \[
\begin{bmatrix}a & b \\
b & -a
\end{bmatrix}
\]
with $a^2 + b^2 = 1$, the values of $a$ and $b$ do not correspond with the values $u_1$ and $u_2$ of the unit vector along the line.
(c) (10 points) For $A$ the reflection matrix as in the previous part, find a matrix $B$ such that

$$AB \neq BA$$  \hspace{1cm} (1)

**Solution:** You need not write the actual components of $B$ if you can describe it as a geometric motion and show the non-equation (1) using a picture.

Almost any matrix will work here, with the notable exceptions of scaling matrices, reflections over the lines $y = x$ and $x = y$, and projection matrices onto the lines $y = x$ and $x = y$. In particular, the matrix obtained in part (a) will not suffice.

We can use the projection matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ representing projection onto the $x$-axis. In this case:

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Clearly, $AB \neq BA$.

**Alternatively:** one can take for example a rotation matrix, say

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and consider $AB$ and $BA$ as the corresponding linear maps of $\mathbb{R}^2$. If one draws a smiley fact, for example, then its image under $AB$ is different than its image under $BA$. 

4. (20 points) Consider the matrix 

\[ A = \begin{pmatrix}
1 & 2 & 1 & 2 \\
3 & 6 & 5 & 4 \\
-1 & -2 & 1 & -4 \\
0 & 0 & -1 & 1 \\
\end{pmatrix} \]

which satisfies 

\[ \text{rref}(A) = \begin{pmatrix}
1 & 2 & 0 & 3 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} \]

(a) (5 points) Find a basis for \( \text{image}(A) \). (You do not need to justify that it is a basis)

**Solution:** A basis is given by the columns of \( A \) which have a leading 1 in \( \text{rref}(A) \). These are the first and third columns. Thus, a basis for the image of \( (A) \) is given by

\[ \begin{pmatrix}
1 \\
3 \\
-1 \\
0 \\
\end{pmatrix} \text{ and } \begin{pmatrix}
1 \\
5 \\
1 \\
-1 \\
\end{pmatrix} \]

A common mistake here was giving the columns of \( \text{rref}(A) \) rather than the columns of \( A \). The columns of \( \text{rref}(A) \) are not even in the image of \( A \) in this case.

(b) (5 points) Express the fourth (right-most) column of \( A \) as a linear combination of the first and third columns.

**Solution:** To find the relation between the fourth column and the first and third columns, we again look at \( \text{rref}(A) \). In \( \text{rref}(A) \) we have

\[(\text{column 4}) = 3 \cdot (\text{column 1}) - (\text{column 3})\]

Thus, the same relation must hold between the columns of \( A \). So the relation we are looking for is

\[ \begin{pmatrix}
2 \\
4 \\
-4 \\
1 \\
\end{pmatrix} = 3 \cdot \begin{pmatrix}
1 \\
3 \\
-1 \\
0 \\
\end{pmatrix} - \begin{pmatrix}
1 \\
5 \\
1 \\
-1 \\
\end{pmatrix} \]

Again, a common mistake here was to write the relation just between the columns of \( \text{rref}(A) \) and not between the columns of \( A \).
(c) (10 points) For $A$ as above, find a basis for ker($A$). Justify that it is a basis by showing that it satisfies the two properties of a basis.

**Solution:** First off, ker($A$) = ker(rref($A$)), so we can just compute the kernel of the reduced row echelon form matrix. For this the equations are

\[
\begin{align*}
    x + 2y + 3w &= 0 \\
z - w &= 0.
\end{align*}
\]

Solving for the pivot variables, $x, z$ in terms of the free variables $y, w$ gives $x = -2y - 3w, z = w$. Thus

\[
\text{ker(rref($A$))} = \left\{ \begin{pmatrix} -2y - 3w \\ y \\ w \\ w \end{pmatrix} : y, w \in \mathbb{R} \right\}
\]

\[
= \left\{ y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix} : y, w \in \mathbb{R} \right\}
\]

\[
= \text{span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.
\]

We claim that the vectors $(-2, 1, 0, 0)^T$ and $(-3, 0, 1, 1)^T$ form a basis. (The $T$ superscript is the transpose, which here just means I’m writing these vectors horizontally instead of vertically). The two defining properties of a basis are:

1. that they span the space. This is clear since we have already shown above that the span of the two vectors equals the ker.
2. that they are linearly independent. To see this, assume

\[
\vec{0} = c \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix},
\]

and note that this implies $0 = c$ (second component) and $0 = d$ (third or fourth component).
5. (20 points)  (a) (4 points) Complete the definition: \( V \) is a linear subspace of \( \mathbb{R}^n \) if... (Hint: there are three (3) conditions)

Solution:

- \( V \) contains the zero vector i.e. \( \vec{0} \in V \).
- \( V \) is closed under addition i.e. for every \( \vec{v}_1, \vec{v}_2 \in V \), \( \vec{v}_1 + \vec{v}_2 \in V \).
- \( V \) is closed under scalar multiplication i.e. for every \( \vec{v} \in V \), and \( k \in \mathbb{R} \) \( k\vec{v} \in V \).

For the next problem you did not need to give reasons, just true or false was fine.

(b) (6 points) True or false, the following are linear subspaces:

- \( W = \{(x, y) : x \leq y \} \)
  False
  The set \( W = \{(x, y) : x \leq y \} \) is not closed under scalar multiplication. If it were a vector space then for every \((x, y) \in W\) and \( k \in \mathbb{R} \) we should have \( k(x, y) \in W \), but for example \((x, y) = (1, 2) \in W\) but taking \( k = -1 \), \(-x, y) = (-1, -2) \not\in W \).

- \( W = \text{span}(\vec{v}_1, \vec{v}_2) \) for vectors \( \vec{v}_1, \vec{v}_2 \) in \( \mathbb{R}^n \).
  True
  We showed this in class.

- \( W = \{(x, y) : x + y = 1 \} \)
  False
  All three axioms of a vector space fail here. For example, the zero vector does not belong to \( W \), because \( \vec{0} = (0, 0) \) and \( 0 + 0 \neq 1 \).
(c) (10 points) Let $V \subset \mathbb{R}^n$, and define the perpendicular space 

$$V^\perp = \{ \vec{x} : \vec{x} \cdot \vec{v} = 0 \text{ for all } \vec{v} \in V \}.$$ 

Verify that $V^\perp$ is a linear subspace by showing that it satisfies the three properties.

**Solution:** We need to show that $V^\perp$ contains $\vec{0}$, that it is closed under addition, and that it is closed under multiplication.

To show that it contains zero, we must, by definition, show that 

$$\vec{0} \cdot \vec{v} = 0$$

for all $\vec{v} \in V$. This is true because the dot product of the zero vector with any vector is zero.

To show that $V^\perp$ is closed under addition, assume that $\vec{x}$ and $\vec{y}$ are in $V^\perp$. We must show that that $\vec{x} + \vec{y} \in V^\perp$. To do this, we check the condition for being a member of $V^\perp$. We must show that for all $v \in V$, $(\vec{x} + \vec{y}) \cdot \vec{v} = 0$. We distribute

$$(\vec{x} + \vec{y}) \cdot \vec{v} = \vec{x} \cdot \vec{v} + \vec{y} \cdot \vec{v}$$

Since $\vec{x}, \vec{y} \in V^\perp$, we have that $\vec{x} \cdot \vec{v} = 0$ and $\vec{y} \cdot \vec{v} = 0$. Thus,

$$(\vec{x} + \vec{y}) \cdot \vec{v} = \vec{x} \cdot \vec{v} + \vec{y} \cdot \vec{v} = 0 + 0 = 0$$

We conclude that

$$\vec{x} + \vec{y} \in V^\perp$$

Finally, to show that $V^\perp$ is closed under scalar multiplication, let $k \in \mathbb{R}$ be a scalar, and let $\vec{x} \in V^\perp$. Since $\vec{x} \in V^\perp$, we have that $\vec{x} \cdot \vec{v} = 0$ for every $\vec{v} \in V$. Then

$$(k\vec{x}) \cdot \vec{v} = k(\vec{x} \cdot \vec{v}) = k \cdot 0 = 0$$

Thus, $k\vec{x} \in V^\perp$, so $V^\perp$ is closed under scalar multiplication.

We have verified all three properties. Thus, $V^\perp$ is a linear subspace of $\mathbb{R}^n$.

**Some common mistakes:** The most common mistakes had to do with not being clear about where each vector involved lives. Expressions like

$$\vec{x} \cdot (\vec{v} + \vec{w})$$

or

$$(\vec{x} + \vec{y}) \cdot (\vec{v} + \vec{w})$$

were common. If $\vec{x}, \vec{y} \in V^\perp$ and $\vec{v}, \vec{w} \in V$ in the above, then these expressions are irrelevant towards showing closure under addition. Similarly, to show that $V^\perp$ contains zero, a common mistake was to write

$$\vec{x} \cdot \vec{v} = 0$$

which, unless you specified that $\vec{x} = 0$, is not relevant.