Attaching a 2-cell

*Compare Proposition 1.26 in Hatcher.*

Attach a 2-cell to $X$ by means of the attaching map $f: S^1 \to X$ to form the space $Y$, so that we have the pushout square of spaces

$$
\begin{array}{ccc}
D^2 & \xrightarrow{g} & Y \\
\uparrow i & & \uparrow j \\
S^1 & \xrightarrow{f} & X
\end{array}
$$

We wish to know the effect on the fundamental group.

Choose a basepoint $d_1 \in S^1$ and put $x_1 = f(d_1)$.

**Theorem 2** Given the above data, we have the pushout square of groups

$$
\begin{array}{ccc}
\pi_1(D^2, d_1) & \xrightarrow{g_*} & \pi_1(Y, x_1) \\
\uparrow i_* & & \uparrow j_* \\
\pi_1(S^1, d_1) & \xrightarrow{f_*} & \pi_1(X, x_1)
\end{array}
$$

Here, $\pi_1(D^2)$ is trivial because $D^2$ is convex and hence contractible, and $\pi_1(S^1) = \mathbb{Z}$, generated by $[\omega_1]$. Moreover, we don’t wish to assume $f$ is based; let $x_0$ be some other point of $X$, with $h$ a path from $x_0$ to $x_1$. By combining with the isomorphisms $\beta_h$ of Proposition 1.5, we deduce the more general version.

**Corollary 3** We have the pushout square of groups

$$
\begin{array}{ccc}
\{1\} & \xrightarrow{} & \pi_1(Y, x_0) \\
\uparrow & & \uparrow j_* \\
\mathbb{Z} & \xrightarrow{} & \pi_1(X, x_0)
\end{array}
$$

where the lower homomorphism takes $1 \in \mathbb{Z}$ to $\alpha = \beta_h(f_*[\omega_1])$.

**Interpretation** By definition of pushouts, given any group $H$ and homomorphisms $\phi_1: \{1\} \to H$ and $\phi_2: \pi_1(X, x_0) \to H$ that agree on $\mathbb{Z}$, there is a unique homomorphism $\phi: \pi_1(Y, x_0) \to H$ that makes the whole diagram commute. Thus $\phi_1$ is trivial and $\phi_2(\alpha) = 1$. Then $\ker \phi_2$ is a normal subgroup of $\pi_1(X, x_0)$ that contains $\alpha$. Let $N$ be the smallest normal subgroup of $\pi_1(X, x_0)$ that contains $\alpha$ (it is the intersection of all such subgroups). Then we may identify $\pi_1(Y, x_0)$ with the quotient group $\pi_1(X, x_0)/N$ and $j_*$ with the natural quotient homomorphism.

**Proof of Theorem** The given pushout square diagram (1) does not lend itself to direct application of van Kampen’s Theorem. The key idea is to *attach a collar*. Define the
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collar \( V = \{ x \in D^2 : \| x \| > 1/2 \} \) of \( S^1 \) in \( D^2 \); it contains \( S^1 \) as a deformation retract.

Now we form the expanded diagram of spaces, which contains diagram (1),

\[
\begin{array}{ccc}
e^2 & \subset & D^2 \\
\downarrow & \subset & \downarrow \\
V - S^1 & \subset & V \\
\downarrow & \subset & \downarrow \\
S^1 & \subset & X
\end{array}
\]

It contains five pushout squares, as pushout squares can be stacked.

Now we can apply van Kampen, taking \( A_1 = e^2 \) and \( A_2 = X \cup g(V) \) and a basepoint \( d_2 \in V - S^1 \) that retracts to \( d_1 \), to obtain the pushout square of groups

\[
\begin{array}{ccc}
\pi_1(e^2, d_2) & \longrightarrow & \pi_1(Y, y_2) \\
\downarrow & \uparrow g_* & \downarrow j_* \\
\pi_1(V - S^1, d_2) & \longrightarrow & \pi_1(X \cup g(V), y_2)
\end{array}
\]

where \( y_2 = g(d_2) \).

This is not quite what we want. The basepoint \( y_2 \) is particularly inconvenient, as it does not lie in \( X \). We may change any of the groups by an isomorphism. The retraction \( V - S^1 \subset V \rightarrow S^1 \) induces an isomorphism \( \pi_1(V - S^1, d_2) \cong \pi_1(S^1, d_1) \). The induced retraction \( s: X \cup g(V) \rightarrow X \) induces an isomorphism \( s_*: \pi_1(X \cup g(V), y_2) \cong \pi_1(X, x_1) \), and the resulting homomorphism \( \pi_1(S^1, d_1) \rightarrow \pi_1(X, x_1) \) is clearly \( f_* \).

Finally, we take \( k \) to be a path from \( x_1 \) to \( y_2 \) such that \( s \circ k \) is the constant path at \( x_1 \) and use the commutative diagram

\[
\begin{array}{ccc}
\pi_1(Y, y_2) & \longrightarrow & \pi_1(Y, x_1) \\
\downarrow & \uparrow j_* & \downarrow j_* \\
\pi_1(X \cup g(V), y_2) & \longrightarrow & \pi_1(X \cup g(V), x_1) \\
\downarrow s_* & \uparrow s_* \\
\pi_1(X, x_1)
\end{array}
\]

to fix up the right side of diagram (4). \( \square \)