The Tangent Vector to a Curve

Let \( C \) be a space curve parametrized by the differentiable vector-valued function \( r(t) \). At the point \( P_0 (= r(t_0)) \) of \( C \), we have the derivative (or velocity) vector

\[
v(t_0) = r'(t_0) = \lim_{h \to 0} \frac{r(t_0 + h) - r(t_0)}{h} = \lim_{h \to 0} \frac{\vec{P}_0 \vec{P}}{h},
\]

where we write \( P \) for the point \( r(t_0 + h) \) on \( C \). We give a direct geometric interpretation of this vector, assuming it is not zero. It follows from equation (1) that for small \( h \not= 0 \), we have \( \vec{P}_0 \vec{P} \not= \vec{P}_0 \vec{P} \), so that \( \vec{P}_0 \vec{P} \) is a nonzero vector.

Theorem 2 Assume that \( r'(t_0) \not= 0 \). Let \( \theta(h) \) be the angle between the vectors \( \vec{P}_0 \vec{P} \) and \( r'(t_0) \) (which is defined for sufficiently small \( h \not= 0 \)). Then:

(a) \( \lim_{h \to 0^+} \theta(h) = 0 \);

(b) \( \lim_{h \to 0^-} \theta(h) = \pi \); or equivalently, the angle between \( \vec{P}_0 \vec{P} \) and \(-r'(t_0)\) tends to 0 as \( h \to 0^- \).

Proof The standard angle formula gives

\[
\cos \theta(h) = \frac{\vec{P}_0 \vec{P} \cdot r'(t_0)}{\|\vec{P}_0 \vec{P}\| \|r'(t_0)\|} = \frac{[r(t_0 + h) - r(t_0)] \cdot r'(t_0)}{\|r(t_0 + h) - r(t_0)\| \|r'(t_0)\|}.
\]

We cannot take the limit directly, because we get \( 0/0 \). However, if we first divide numerator and denominator by \( h \) and assume \( h > 0 \), we can apply equation (1) directly (using the continuity of norms and dot products) to get

\[
\cos \theta(h) = \frac{r(t_0 + h) - r(t_0) \cdot r'(t_0)}{\|r(t_0 + h) - r(t_0)\| \|r'(t_0)\|} \quad \longrightarrow \quad \frac{r'(t_0) \cdot r'(t_0)}{\|r'(t_0)\| \|r'(t_0)\|} = 1
\]

as \( h \to 0^+ \). Finally, we apply the continuous function \( \cos^{-1} \) to deduce that

\[
\theta(h) = \cos^{-1}(\cos \theta(h)) \longrightarrow \cos^{-1}(1) = 0 \quad \text{as} \quad h \to 0^+,
\]

as required.

If \( h < 0 \), equation (3) is off by a sign, and we get \( \theta \to \cos^{-1}(-1) = \pi \) instead. \( \square \)

Geometrically, the vector \( r'(t_0) \) is tangent to the curve \( C \) at \( P_0 \). This leads to the following definition.

Definition 4 The tangent line to \( C \) at \( P_0 \) is the line through \( P_0 \) in the direction of the vector \( r'(t_0) \).

Thus its parametric equation (with parameter \( u \)) is (see (13.3.2))

\[
R(u) = r(t_0) + ur'(t_0).
\]