Derivatives

Given a function \( f \), assume that \( f'(a) = m \), where \( a \) is fixed. We put
\[
\epsilon(h) = \frac{f(a + h) - f(a)}{h} - m \quad \text{for } h \neq 0
\]
in order to arrange \( \epsilon(h) \to 0 \) as \( h \to 0 \). When we multiply up, we get
\[
f(a + h) = f(a) + mh + h\epsilon(h)
\]
On the right, we have the constant term \( f(a) \), then a constant multiple of \( h \), then an error term which is small compared to \( h \).

Definition 2 The function \( f \) is differentiable at \( a \), with derivative \( f'(a) = m \), if and only if equation (1) holds with \( \epsilon(h) \to 0 \) as \( h \to 0 \).

In equation (1) we no longer have any division by \( h \), and it remains valid for \( h = 0 \), whatever value we choose for \( \epsilon(0) \). The reasonable choice is \( \epsilon(0) = 0 \), to make the function \( \epsilon \) continuous at \( h = 0 \). This form of the definition leads to cleaner proofs of the standard rules.

Theorem 3 If \( f \) is differentiable at \( a \), it is continuous at \( a \).  

Proof We must show that \( f(a+h) \to f(a) \) as \( h \to 0 \). From equation (1) we have
\[
f(a + h) \to f(a) + 0 + 0 = f(a) \quad \text{as } h \to 0. \quad \square
\]

Constant multiples If we multiply equation (1) by a constant \( c \), we get
\[
 cf(a + h) = cf(a) + cmh + c\epsilon(h)
\]
to which Definition 2 applies immediately. We write \( cm = cf'(a) \).

Theorem 4 Given a constant \( c \), if \( f \) is differentiable at \( a \), then so is \( cf \), with derivative \( cf'(a) \). \( \square \)

Sums and Differences Assume also that \( g'(a) = p \), so that like equation (1) we have
\[
g(a + h) = g(a) + ph + h\theta(h)
\]
with \( \theta(h) \to 0 \) as \( h \to 0 \). If we add (or subtract) equations (1) and (5), we see that
\[
f(a + h) \pm g(a + h) = f(a) \pm g(a) + (m \pm p)h + h\{\epsilon(h) \pm \theta(h)\}
\]
which has the form prescribed by Definition 2 for showing that \( f \pm g \) is differentiable.

Theorem 6 If \( f \) and \( g \) are differentiable at \( a \), then so is \( f \pm g \), with derivative \( f'(a) \pm g'(a) \). \( \square \)

The Product Rule We can just as well multiply equations (1) and (5). There are nine terms, which we collect as we go,
\[
f(a + h)g(a + h) = f(a)g(a) + \{f(a)p + mg(a)\}h \\
h\{f(a)\theta(h) + mph + m\theta(h) + \epsilon(h)g(a) + \epsilon(h)ph + \epsilon(h)\theta(h)\}
\]
Again, Definition 2 applies, and we plug in \( p = g'(a) \) and \( m = f'(a) \).
Theorem 7  If $f$ and $g$ are differentiable at $a$, then so is $fg$, with derivative

$$f'(a)g(a) + f(a)g'(a)$$

The Chain Rule  Suppose that $u = g(x)$ and $y = f(u)$, so that $y = f(u) = f(g(x))$. Assume first that $g'(a) = p$, so that we have by equation (5)

$$g(a + h) = g(a) + ph + h\theta(h)$$

$$= b + k$$

where we write $b = g(a)$ and $k = h(p + \theta(h))$ in preparation for the next step, which is to expand $f(g(a+h)) = f(b+k)$. We note that $k \to 0$ as $h \to 0$.

Assume that $f''(b) = q$, so that like equation (1),

$$f(g(a+h)) = f(b+k) = f(b) + qk + k\epsilon(k)$$

$$= f(b) + qph + h\{q\theta(h) + p\epsilon(k) + \theta(h)\epsilon(k)\}$$

where $\epsilon(k) \to 0$ as $k \to 0$ and we substituted for $k$ in some places. Once again, Definition 2 applies, and the derivative of $f \circ g$ is $qp$. (It is possible to have $k = 0$ exactly, but this is not a problem if we define $\epsilon(0) = 0$ as suggested, to make $\epsilon$ continuous.) We plug in $q = f'(b) = f'(g(a))$ and $p = g'(a)$.

Theorem 8  If $g$ is differentiable at $a$ and $f$ is differentiable at $g(a)$, then the composite $f \circ g$ is differentiable at $a$, with derivative $f'(g(a))g'(a)$. □

Reciprocals  We apply Theorem 8 with $f(u) = 1/u$, for which $f'(u) = -1/u^2$.

Theorem 9  If $f$ is differentiable at $a$, then so is $1/f$, with derivative $-f'(a)/f(a)^2$ (provided that $f(a) \neq 0$). □

The Quotient Rule

Theorem 10  If $u$ and $v$ are differentiable at $a$, then so is $u/v$, with derivative

$$\frac{v(a)u'(a) - u(a)v'(a)}{v(a)^2}$$

provided $v(a) \neq 0$.

Proof  We write $u/v = u.(1/v)$ and apply the product rule Theorem 7 and Theorem 9, to get

$$u'(a)\frac{1}{v(a)} + u(a)\frac{-v'(a)}{v(a)^2}$$

which can be rearranged as stated. □