Relations between Points and Subsets

Assume given a metric or topological space $X$ and any subset $A \subset X$. We discuss the possible relationships between a point $x \in X$ and $A$ in terms of neighborhoods.

First Level We note that every neighborhood $N$ of $x$ contains $x$ (so is non-empty). At this level, we make no distinction between $x$ and other points of $N$. This provides enough information for most of our work.

There are exactly three possibilities, with no overlap:

Case 1. $x$ has a neighborhood $N$ that is contained in $A$. We call $x$ an interior point of $A$. Such points form the interior set $\text{Int} A$, or sometimes $\breve{A}$, of $A$. Since $x \in N \subset A$, we have $\text{Int} A \subset A$.

Case 2. $x$ has a neighborhood $N$ that is contained in the complement $X - A$ of $A$. We call $x$ an exterior point of $A$. Since $x \in N$, these points never lie in $A$. They form the exterior set $\text{Ext} A = \text{Int}(X - A)$ of $A$.

Case 3. Otherwise, every neighborhood $N$ of $x$ contains both a point of $A$ and a point of $X - A$. We call $x$ a frontier point or boundary point of $A$. Such points form the frontier or boundary set, $\text{Fr} A$ or $\text{Bd} A$, of $A$. These points may or may not lie in $A$. By symmetry, $\text{Fr}(X - A) = \text{Fr} A$.

To summarize, every point $x \in X$ lies in exactly one of the three sets $\text{Int} A$, $\text{Ext} A$, and $\text{Fr} A$. The ambiguity in Case 3 motivates the next definition.

Definition 1 We call $A$ closed (in $X$) if it contains all of its boundary points. We call $A$ open (in $X$) if it contains none of its boundary points.

It is obvious from the symmetry that $A$ is open if and only if $X - A$ is closed.

Lemma 2 For any subset $A \subset X$:

(a) The interior $\text{Int} A$ is open;

(b) If $V \subset A$ is open in $X$, then $V \subset \text{Int} A$.

Thus $\text{Fr} A = X - (\text{Int} A \cup \text{Ext} A) = X - (\text{Int} A \cup \text{Int}(X - A))$ is closed.

The closure $\text{Cl} A = \overline{A}$ of $A$ may be defined as $\text{Int} A \cup \text{Fr} A$ or as $A \cup \text{Fr} A$. Since $\text{Cl} A = X - \text{Ext} A$, it is closed, and if $F$ is any closed set that contains $A$, i.e. $F \supset A$, we must have $\text{Cl} A \subset F$.

Second Level This is more subtle. At this level, we do distinguish between $x$ and other points of $N$. The limit points and isolated points of $A$ can now be defined. There are now eight possibilities, summarized in the table:

<table>
<thead>
<tr>
<th>Case</th>
<th>Description</th>
<th>$x \in A$</th>
<th>$x \notin A$</th>
<th>in $A$?</th>
<th>int?</th>
<th>limit point?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$x$ has a neighborhood $N$ with $N - x$ empty.</td>
<td>$x \in A$</td>
<td>$x \notin A$</td>
<td>Int</td>
<td>isolated</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>$x$ has a neighborhood $N$ with non-empty $N - x \subset A$.</td>
<td>$x \in A$</td>
<td>$x \notin A$</td>
<td>Ext</td>
<td>exterior</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>$x$ has a neighborhood $N$ with non-empty $N - x \subset X - A$.</td>
<td>$x \in A$</td>
<td>$x \notin A$</td>
<td>Fr</td>
<td>limit</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>None of the above: every $N - x$ contains points of $A$ and of $X - A$.</td>
<td>$x \in A$</td>
<td>$x \notin A$</td>
<td>Fr</td>
<td>limit</td>
<td></td>
</tr>
</tbody>
</table>