Math 202 Practice exam 2

Show all work in a clear and concise manner to get maximum credit. Circle your answers.

1. Consider a wire in space parametrized by
   \[ \vec{c}(t) = \langle 3t, -t^2, 2t^2 \rangle, \quad 0 \leq t \leq 1. \]
   a. (10 points) Set up, but do not evaluate, the integral for the arc length of the wire.

   \[ \vec{c}'(t) = \langle 3, -2t, 4t \rangle, \quad |\vec{c}'(t)| = \sqrt{9 + 20t^2}, \quad ds = \sqrt{9 + 20t^2} \, dt, \quad L = \int_0^1 \sqrt{9 + 20t^2} \, dt. \]

   b. (10 points) Suppose the density of the wire at a point \((x, y, z)\) is \(\rho(x, y, z) = x\).

   Find the mass of the wire.

   \[ M = \int_0^1 \rho(x(t), y(t), z(t)) \, ds = \int_0^1 3t\sqrt{9 + 20t^2} \, dt = \frac{3}{40} \left[ (9+20t^2)^{3/2} \right]_0^1 = \frac{1}{20}((29)^{3/2} - 27). \]

2. (15 points) Let \(\vec{F}(x, y, z) = \langle e^x yz, e^x z + 2yz, e^x y + y^2 + 1 \rangle\).

   Evaluate \(\int_{\vec{c}} \vec{F} \cdot d\vec{s}\) where \(\vec{c}(t) = \langle \cos t, \sin t, t \rangle, \quad 0 \leq t \leq \frac{\pi}{2}\).

   Hint: look for a shortcut.

   \(\vec{F}\) is conservative, i.e., \(\phi = e^x yz + y^2 z + z\) is a potential function for \(\vec{F}\) so

   \[ \int_{\vec{c}} \vec{F} \cdot d\vec{s} = \phi(\vec{c}(\frac{\pi}{2})) - \phi(\vec{c}(0)) = \phi(0, 1, \frac{\pi}{2}) - \phi(1, 0, 0) = \frac{3\pi}{2}. \]

3. Consider the double integral \(\int_0^1 (\int_x^\sqrt{x} e^{\frac{y}{x}} dy) dx\).

   a. (5 points) Sketch the region \(D\) of integration.

   b. (10 points) Reverse the order of integration and evaluate.

   \(D\) is bounded by the graphs \(y = \sqrt{x}\) (top) and \(y = x\) (bottom), \(0 \leq x \leq 1\) so when reversing the order of integration, the graph \(x = y^2\) is now the “bottom”.

   \( \int_0^1 (\int_x^\sqrt{x} e^{\frac{y}{x}} dy) dx = \int_0^1 (\int_{y^2}^y e^{\frac{y}{x}} dx) dy = \int_0^1 (ye^{\frac{y}{y^2}}) dy = \int_0^1 y(e - e^y) dy = \left( \frac{e}{2} y^2 - ye^y + e^y \right) |_0^1 = \frac{e}{2} - 1. \)
4. Let C be the boundary of the triangle with vertices (0,0), (2,0), (1,1) oriented in the counterclockwise direction.

a. (10 points) Evaluate \( \int_C xy \, dx + x^2 \, dy \) (do not use Green’s theorem).

The boundary C is made up of the lines \( y = 0 \), \( y = 2 - x \), \( y = x \). Hence

\[
\int_C xy \, dx + x^2 \, dy = -\int_1^2 (x(2-x) - x^2) \, dx - \int_0^1 (x^2 + x^2) \, dx
\]

\[
= \int_1^2 (2x^2 - 2x) \, dx - \int_0^1 2x^2 \, dx = (\frac{2}{3} x^3 - x^2)|_1^2 - \frac{2}{3} = (\frac{16}{3} - 4 - \frac{2}{3} + 1) - \frac{2}{3} = 1.
\]

b. (10 points) Verify your answer by using Green’s theorem to compute the line integral of part a.

\[
P(x,y) = xy, \ Q(x,y) = x^2 \text{ so } \int_C P \, dx + Q \, dy = \int \int_T (Q_x - P_y) \, dA = \int \int x \, dA = \int_0^1 \int_{2-y}^{2-y} x \, dx \, dy = \int_0^1 (2 - 2y) \, dy = 2 - 1 = 1.
\]

5. (15 points) Let \( f(x,y) = e^{x+y} \) and let R be the parallelogram in the xy plane bounded by the lines \( x - 2y = 0 \), \( x - 2y = 3 \), \( 2x - y = 0 \), \( 2x - y = 3 \). Use the linear change of variables \( u = x - 2y, \ v = 2x - y \) to evaluate \( \int \int_R f(x,y) \, dA \). Hint: Find the domain \( R^* \) in the uv plane corresponding to R and also \( x = x(u,v), \ y = y(u,v) \).

The image of R under the linear transformation \( u = x - 2y, \ v = 2x - y \) is the square \( R^* = \{(u,v) : 0 \leq u \leq 3, 0 \leq v \leq 3 \} \). We need to solve for the inverse transformation \( x = x(u,v), \ y = y(u,v) \) that maps \( R^* \) in a one to one way onto R. This is easily done: \( x = \frac{1}{3}(-u + 2v), \ y = \frac{1}{3}(-2u + v) \). This allows us to compute the Jacobian determinant

\[
\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}
-\frac{1}{3} & \frac{2}{3} \\
-\frac{2}{3} & \frac{1}{3}
\end{vmatrix} = \frac{1}{3}.
\]

Hence,

\[
\int \int_R e^{x+y} \, dA = \int \int_{R^*} e^{(-u+v)} \frac{1}{3} \, dudv = \frac{1}{3} \int_0^3 \int_0^3 e^{v-u} \, dudv = \frac{1}{3} (1 - e^{-3})(e^3 - 1).
\]

6. (15 points) Use double integrals to find the volume of the region inside the cylinder \( x^2 + y^2 = 2\pi \), outside the cylinder \( x^2 + y^2 = \frac{3\pi}{2} \), above the plane \( z = 0 \) and below the
surface $z = \cos(x^2 + y^2)$. Hint: Use polar coordinates.

The elementary region $W$ in three space is bounded on top by the graph $z = \cos(x^2 + y^2)$ and on bottom by the graph $z=0$ over the domain $D = \{(x, y) : \frac{3\pi}{2} \leq x^2 + y^2 \leq 2\pi\}$. Hence

$$ V = \int \int_D \cos(x^2 + y^2) dA. $$

In polar coordinates, $D$ corresponds to $D^* = \{(r, \theta) : \sqrt{\frac{3\pi}{2}} \leq r \leq \sqrt{2\pi}, 0 \leq \theta \leq 2\pi\}$ so

$$ V = \int \int_{D^*} \cos(r^2) r dr d\theta = \int_0^{2\pi} \left( \int_{\sqrt{\frac{3\pi}{2}}}^{\sqrt{2\pi}} \cos(r^2) r dr \right) d\theta = \pi \cos(r^2) \bigg|_{\sqrt{\frac{3\pi}{2}}}^{\sqrt{2\pi}} = \pi (1 - 0) = \pi. $$