Strictly monotone functions and the inverse function theorem

We have seen that for a monotone function \( f : (a, b) \rightarrow \mathbb{R} \), the left and right hand limits
\[
y_0^- = \lim_{x \to x_0^-} f(x) \quad \text{and} \quad y_0^+ = \lim_{x \to x_0^+} f(x)
\]
both exist for all \( x_0 \in (a, b) \). This implies any discontinuity of \( f \) is a jump and there are at most a countable number. Obviously if \( f(x) \) is continuous on \( I = [a, b] \) then the image \( f(I) \) is an interval \([f(a), f(b)]\) (a point if \( f \) is a constant). However the converse is also true if \( f \) is monotone.

**Theorem 0.1.** Let \( f : I \rightarrow \mathbb{R} \) be monotone increasing with range an interval. Then \( f \) is continuous on \((a,b)\)

**Proof.** Suppose for contradiction that \( f \) has a jump at \( x_0 \). Then at least one of the intervals \((y_0^-, f(x_0))\) or \((f(x_0), y_0^+)\), must be nonempty. Pick one and call it \( J \) and note that \( J \subset (y_0^-, y_0^+)\) so the image of \( f \) is not an interval.

**Theorem 0.2.** Let \( f \) be continuous and one to one on an interval \((a,b)\), then \( f \) is either strictly decreasing or strictly increasing.

**Proof.** Let \( a < x_1 < x_2 < b \). Then either \( f(x_2) > f(x_1) \) or \( f(x_2) < f(x_1) \) Suppose the first possibility; then we claim \( f \) is strictly increasing on \((a,b)\). Let \( a < x'_1 < x'_2 < b \) be any other ordered two points in the interval. Set \( x(t) = tx'_1 + (1-t)x_1, y(t) = tx'_2 + (1-t)x_2 \). Then \( a < x(t) < y(t) < b \) for \( 0 \leq t \leq 1 \). Set \( g(t) = f(y(t)) - f(x(t)) \). Then \( g \) is the composition of continuous functions so is continuous on \([0,1]\). Also \( g(0) \neq 0 \) since \( f \) is one to one so \( g(t) \) cannot change sign by the Intermediate value theorem. Since \( g(0) = f(x_2) - f(x_1) > 0 \), \( g(t) > 0 \) and hence \( g(1) = f(x'_2) - f(x'_1) > 0 \). The second possibility follows by a similar argument.

**Theorem 0.3.** If \( f \) is continuous and one to one on an interval, then \( f^{-1} \) is also continuous.

**Proof.** By the previous theorem, \( f \) is either strictly increasing or strictly decreasing. Suppose the former and let \( x_0 \) be in the interval with \( y_0 = f(x_0) \). We must show \( \lim_{y \to y_0} f^{-1}(y) = x_0 \).

Let \( \varepsilon > 0 \) be given. If \( x_0 - \varepsilon < x_0 < x_0 + \varepsilon \), then \( f(x_0 - \varepsilon) < f(x_0) < f(x_0 + \varepsilon) \). Choose
\[
\delta = \min \left( f(x_0) - f(x_0 - \varepsilon), f(x_0 + \varepsilon) - f(x_0) \right).
\]
Then \( f(x_0 - \varepsilon) < f(x_0) - \delta \) and \( f(x_0) + \delta < f(x_0 + \varepsilon) \). Hence if \( f(x_0) - \delta < y < f(x_0) + \delta \), then \( f(x_0 - \varepsilon) < y < f(x_0 + \varepsilon) \). Since \( f \) is strictly increasing, so is \( f^{-1} \) and therefore \( x_0 - \varepsilon < f^{-1}(y) < x_0 + \varepsilon \). We have shown \( |f^{-1}(y) - f^{-1}(y_0)| < \varepsilon \) if \( |y - y_0| < \delta \) which is what we needed to show.

\[1\]
**Theorem 0.4.** (Inverse function) Let $f$ be a strictly monotone continuous function on $[a, b]$ with $f$ differentiable at $x_0 \in (a, b)$ and $f'(x_0) \neq 0$. Then $f^{-1}$ exists and is continuous and strictly monotone. Moreover, $f^{-1}$ is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$ 

**Proof.** Since $f$ is strictly monotone and continuous, $f$ is one to one onto its range which is the interval $J = [f(a), f(b)]$. The inverse function $g = f^{-1}$ is also strictly increasing mapping $J$ onto $I$ so $g$ is continuous by Theorem 1. Let $y_0 = f(x_0)$. Then

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{g(y) - g(y_0)}{f(g(y)) - f(g(y_0))} = \lim_{y \to y_0} \frac{1}{f(g(y)) - f(g(y_0))}.$$ 

Note that the denominators are not zero. Since $f$ is differentiable at $x_0$ and $g$ is continuous (so as $y \to y_0$, $x \to x_0$)

$$\frac{f(g(y)) - f(g(y_0))}{g(y) - g(y_0)} = f'(x_0) + o(1) \quad \text{as} \quad y \to y_0.$$ 

Therefore,

$$g'(y_0) = \frac{1}{f'(x_0)}.$$

$\square$