MATH 3210-2. Decimal expansion of real numbers.

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In this handout I describe decimal expansion of the real numbers. We start by discussing real numbers in the interval \([0, 10)\).

A *decimal sequence* is a sequence \((x_n)\) of rational numbers satisfying the following property:

\[ x_0 \in \mathbb{Z} \cap [0, 10) \text{ and } x_{n+1} = x_n + a_{n+1}10^{-(n+1)} \text{ where } a_{n+1} \in \mathbb{Z} \cap [0, 10). \]

In other words,

\[ x_n = \sum_{i=0}^{n} a_i \cdot 10^{-i} \]

where \(a_i \in \mathbb{Z} \cap [0, 10)\) for each \(i \in \mathbb{N}\). It is clear that this sequence is increasing.

Let \(x\) be a real number which belongs to the interval \([0, 10)\). The *decimal expansion* of \(x\), is a decimal sequence \((x_n)\) which converges to \(x\).

**Theorem 1.** (1) Each decimal sequence \((x_n)\) converges to a real number \(x \in [0, 10]\). The limit of this sequence belongs to \([0, 10)\) unless \(a_i = 9\) for all \(i \in \mathbb{N}\).

(2) Each real number \(x \in [0, 10)\) has a decimal expansion.

**Proof.** (1) Since the decimal sequence \((x_n)\) is increasing we have

\[ \lim_{n \to \infty} x_n = x = \sup\{x_n | n \in \mathbb{N}\}. \]

Let us check that this supremum belongs to \([0, 10]\).

\[ x_n = \sum_{i=0}^{n} a_i \cdot 10^{-i} \leq \sum_{i=0}^{n} 9 \cdot 10^{-i} = 9 \cdot \sum_{i=0}^{n} 0.1^i = 9 \cdot \frac{1 - 0.1^{n+1}}{1 - 0.1} = 10(1 - 0.1^{n+1}) < 10. \]

Hence 10 is an upper bound for \(\{x_n | n \in \mathbb{N}\}\) which implies that \(x = \sup\{x_n | n \in \mathbb{N}\} \leq 10\). On the other hand, \(x_1 \geq 0\), hence \(x = \sup\{x_n | n \in \mathbb{N}\} \geq x_1 \geq 0\). Thus \(x = \lim_{n \to \infty} x_n \in [0, 10]\).
Suppose that \( a_i < 9 \) for some \( i = k \in \mathbb{N} \). Then for each \( n \geq k \),
\[
\sum_{i=0}^{n} a_i \cdot 10^{-i} \leq \sum_{i=0}^{n} 9 \cdot 10^{-i} + (a_k - 9)10^{-k} = 10(1 - 0.1^{n+1}) - (9 - a_k)10^{-k} \leq 10 - (9 - a_k)10^{-k}.
\]

Note that \( \delta = (9 - a_k)10^{-k} > 0 \). Thus \( x \leq 10 - (9 - a_k)10^{-k} = 10 - \delta < 10 \). Hence \( x \neq 10 \) which means that \( x \in [0, 10) \).

This concludes the proof of (1).

Proof of (2). Given \( x \in [0, 10) \) we will construct \( x_n \) and \( a_n \in \{0, \ldots, 9\} \) so that \( x_n \leq x < x_n + 10^{-n} \) and \( x_{n+1} = x_n + a_{n+1}10^{-(n+1)} \), which means that
\[
x_n = \sum_{i=0}^{n} a_i \cdot 10^{-i}.
\]

For this construction we use the induction on \( n \). Let \( a_0 = x_0 = [x] \), which is the largest integer not exceeding \( x \).

Since \( x \in [0, 10) \), \( a_0 \in \mathbb{Z} \cap [0, 10) \). It is clear that \( x_0 \leq x < x_0 + 1 \).

**Step of the induction.** Suppose that we already have \( x_n \) so that \( x_n \leq x < x_n + 10^{-n} \). The interval \([x_n, x_n + 10^{-n}]\) has the length \( 10^{-n} \). Let \( x_{n+1} \in [x_n, x_n + 10^{-n}] \) be the largest number of the form \( x_n + a10^{-(n+1)} \) which does not exceed \( x \) (where \( a \in \{0, \ldots, 9\} \)). Such maximum exists since there are only ten numbers of such form. Then
\[
x_{n+1} = x_n + a_{n+1}10^{-(n+1)}
\]
and
\[
x_{n+1} = x_n + a_{n+1}10^{-(n+1)} \leq x < x_n + (a_{n+1}+1)10^{-(n+1)} = x_{n+1} + 10^{-(n+1)}.
\]

Thus \( x_{n+1} = x_n + a_{n+1}10^{-(n+1)} \) and \( x \in [x_{n+1}, x_{n+1} + 10^{-(n+1)}] \). Therefore we are done (by induction) constructing \( (x_n) \). Now let’s check that \( (x_n) \) converges to \( x \). The sequence \( (x_n) \) is increasing and is bounded (from above) by \( x \), hence it has a limit \( y \in \mathbb{R} \). Since
\[
x_n \leq x < x_n + 10^{-n}
\]
then (by the squeeze lemma)
\[
y = \lim_{n \to \infty} x_n \leq x \leq \lim_{n \to \infty} (x_n + 10^{-n}) = y + 0
\]
(since \( \lim_{n \to \infty} 10^{-n} = 0 \)). Thus \( y = x \) and we proved that \( \lim_{n \to \infty} x_n = x \). Therefore \( (x_n) \) is a decimal expansion of \( x \). This proves the second assertion of Theorem. \( \square \)

**Remark 2.** You can notice that the proof is very similar to the proof of the Bolzano-Weierstrass theorem (on existence of convergent subsequences in bounded sequences of real numbers).

**Theorem 3.** Suppose that \( (x_n), (y_n) \) are two distinct decimal sequences:
\[
x_n = \sum_{i=0}^{n} a_i \cdot 10^{-i}, \quad y_n = \sum_{i=0}^{n} b_i \cdot 10^{-i}
\]

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so that
\[ \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n. \]
Then there exists \( k \in \mathbb{N} \) so that (after interchanging if necessary the symbols \( x \) and \( y \), \( a \) and \( b \)) we have:

(i) \( a_i = b_i \) for all \( i < k \).

(ii) \( b_k + 1 = a_k \) and \( a_j = 0, b_j = 9 \) for all \( j > k + 1 \).

In other words, the sequences \((a_n), (b_n)\) look like:

\[ a_0, \ldots, a_{k-1}, a_k, 0, \ldots, 0, \ldots \]

\[ a_0, \ldots, a_{k-1}, a_k - 1, 9, \ldots, 9, \ldots \]

respectively.

Proof. Since the sequences \((x_n), (y_n)\) are distinct, there exists a number \( k \in \mathbb{N} \) equal to \( \inf \{ i \in \mathbb{N} | a_i \neq b_i \} \). Then \( a_i = b_i \) for \( i < k \). I will assume that \( a_k > b_k \) (otherwise interchange the symbols \( x \) and \( y \), \( a \) and \( b \)). Suppose that \( a_k > b_k + 1 \). Then for each \( n \geq k + 1 \)

\[ x_n = \sum_{i=0}^{n} a_i \cdot 10^{-i} \geq \sum_{i=0}^{k-1} a_i \cdot 10^{-i} + a_k 10^{-k}, \]

\[ y_n = \sum_{i=0}^{n} b_i \cdot 10^{-i} \leq \sum_{i=0}^{k-1} a_i \cdot 10^{-i} + b_k 10^{-k} + \sum_{i=k+1}^{n} 9 \cdot 10^{-i} \leq \sum_{i=0}^{k-1} a_i \cdot 10^{-i} + b_k 10^{-k} + 10^{-k} = \sum_{i=0}^{k-1} a_i \cdot 10^{-i} + (b_k + 1) 10^{-k}. \]

Thus

\[ \lim_{n \to \infty} x_n \geq \sum_{i=0}^{k-1} a_i \cdot 10^{-i} + a_k 10^{-k} > \sum_{i=0}^{k-1} a_i \cdot 10^{-i} + (b_k + 1) 10^{-k} \geq \lim_{n \to \infty} y_n. \]

This contradicts the assumption that \((x_n)\) and \((y_n)\) have the same limit.

Thus \( b_k = a_k - 1 \). Suppose that \( a_m \neq 0 \) for some \( m > k \). Then by the same argument as above, for all \( n \geq m \) we have

\[ x_n = \sum_{i=0}^{n} a_i \cdot 10^{-i} \geq \sum_{i=0}^{k} a_i \cdot 10^{-i} + a_m 10^{-m} = x_k + a_m 10^{-m}, \]

\[ y_n = \sum_{i=0}^{n} b_i \cdot 10^{-i} \leq \sum_{i=0}^{k-1} a_i \cdot 10^{-i} + (a_k - 1) 10^{-k} + \sum_{i=k+1}^{n} 9 \cdot 10^{-i} \leq \sum_{i=0}^{k-1} a_i \cdot 10^{-i} + (a_k - 1) 10^{-k} + 10^{-k} = \sum_{i=0}^{k} a_i \cdot 10^{-i} = x_k. \]

Hence \( \lim_{n \to \infty} y_n \leq x_k < x_k + a_m 10^{-m} \leq \lim_{n \to \infty} x_n \) which contradicts the assumption that \((x_n)\) and \((y_n)\) have the same limit.
Thus \( a_i = 0 \) for all \( i > k \). This means that
\[
x = x_k = \sum_{i=0}^{k} a_i \cdot 10^{-i}.
\]

Assume that \( b_s < 9 \) for some \( s \geq k + 1 \). Then (similarly to the proof of Theorem 1) for each \( n \geq s \),
\[
y_n = \sum_{i=0}^{n} b_i \cdot 10^{-i} \leq \sum_{i=0}^{k} b_i \cdot 10^{-i} + \sum_{i=k+1}^{n} 9 \cdot 10^{-i} + (b_s - 9)10^{-s}
\]
\[
\leq \sum_{i=0}^{k-1} a_i \cdot 10^{-i} + (a_k - 1)10^{-k} + 10^{-k} - (9 - b_s)10^{-s} =
\]
\[
x_k - 10^{-k} - (9 - b_s)10^{-s} = x - (9 - b_s)10^{-s} = x - \delta,
\]
where \( \delta = (9 - b_s)10^{-s} > 0 \). Thus \( \lim_{n \to \infty} y_n \leq x - \delta < x \). Contradiction. Thus all the assertions of Theorem are proven. \( \square \)

We proved that each decimal sequence converges to a real number in \([0, 10)\), each real number in \([0, 10)\) has a decimal expansion and this expansion is “essentially” unique: the only distinct decimal expansions of the same number have the form:

\[
a_0, \ldots, a_{k-1}, a_k, 0, \ldots, 0, \ldots
\]
\[
a_0, \ldots, a_{k-1}, a_k - 1, 0, \ldots, 0, \ldots
\]

How to define decimal expansions for the real numbers which do not belong to the interval \([0, 10)\)? Given any positive \( x \in \mathbb{R} \) there exists \( m \in \mathbb{N} \) so that \( 10^{m-1} > x \). Thus \( 0 < y = 10^{-m}x < 10 \). Take the decimal expansion \((y_n)\) for \( y \) and define \((x_n = 10^m y_n)\) to be the decimal expansion of \( x \). If \( x < 0 \) then take \( z = -x \). The number \( z \) has decimal expansion \((z_n)\), then define \((x_n = -z_n)\) to be the decimal expansion of \( x \). I leave it for you to verify that the decimal expansion defined this way does not depend on \( m \).

It follows from the Limit Theorem 2.2 that the limit of the decimal expansion \((x_n)\) of \( x \) defined this way converges to \( x \) and the sequence \((x_n)\) has the property similar to the property of the decimal sequences. Namely, for some \( m \in \mathbb{N} \) and \( \epsilon \in \{1, -1\}:\)
\[
x_0 = \epsilon \cdot a_0 10^m
\]
and
\[
x_{n+1} = x_n + \epsilon \cdot a_{n+1} 10^{m-(n+1)},
\]
where \( a_i \in \{0, \ldots, 9\} \). As in the case of the decimal sequences described before, every such generalized decimal sequence converges to a real number and the decimal expansion of any real number is “essentially” unique.