Problem Set 3: Due November 16

1. Solve using the method of characteristics:
   a. $x \ u_x + y \ u_y = 2u, \ u(x, 1) = g(x)$.
   b. $u \ u_x + u_y = 1, \ u(x, x) = \frac{1}{2} x$.
   c. $x \ u_x + 2y \ u_y = 3u, \ u(x, y, 0) = g(x, y)$.

2. Let $w(t)$ satisfy $w'' = -f(t)$, on $(a, b)$, $w(a) = w(b) = 0$.
   a. Show that $w$ is given by
      
      \[ w(t) = \frac{1}{b-a} \{(b-t) \int_a^t (s-a)f(s) \, ds + (t-a) \int_t^b (b-s)f(s) \, ds \} . \]
   
   b. Deduce that the solution of the eigenvalue problem $u'' + \lambda u = 0$, on $(a, b)$, $u(a) = u(b) = 0$ may be found as a solution of the integral equation eigenvalue problem
      
      \[ \mu u(t) = \int_a^b K(s,t)u(s) \, ds , \]
      
      where $\mu = \frac{1}{\lambda}$ and the kernel $K(s,t)$ is given by
      
      \[ K(s,t) = \frac{1}{b-a} \left\{ \begin{array}{ll}
      (s-a)(b-t) & \text{if } a \le s \le t \\
      (t-a)(b-s) & \text{if } t \le s \le b
      \end{array} \right. \]

3. Let $D$ be a domain in the plane and consider the admissible class $A$ of (say) $C^1$ functions in $D$ which vanish on the boundary. Define the Rayleigh quotient
      
      \[ I(u) = \frac{\int_D |\nabla u|^2 \, dxdy}{\int_D u^2 \, dxdy} . \]
      
   a. Suppose that $u(x,y)$ is a smooth minimizer of $I$. Show that $u$ satisfies the Euler-Lagrange equation
      
      \[ \Delta u + \lambda u = 0 , \]
      
      where $\lambda = I(u)$. The constant $\lambda$ is called the principle eigenvalue of the Laplace operator (with Dirichlet boundary conditions) and $u$ is the corresponding eigenfunction. It is in fact true that $u$ cannot change sign. Can you suggest a variational reason for this?
   b. When $D$ is a rectangle, explicitly compute $\lambda$ and $u(x,y)$. Hint: Use separation of variables and solve the Euler-Lagrange equations.