A NOTE ON STARSHAPED COMPACT HYPERSURFACES WITH PRESCRIBED SCALAR CURVATURE IN SPACE FORMS

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ABSTRACT. In [7], Guan, Ren and Wang obtained a $C^2$ a priori estimate for admissible 2-convex hypersurfaces satisfying the Weingarten curvature equation $\sigma_2(\kappa(X)) = f(X, \nu(X))$. In this note, we give a simpler proof of this result, and extend it to space forms.

1. Introduction

In [7], Guan, Ren and Wang solved the long standing problem of obtaining global $C^2$ estimates for a closed convex hypersurface $M \subset \mathbb{R}^{n+1}$ of prescribed kth elementary symmetric function of curvature in general form:

$$\sigma_k(\kappa(X)) = f(X, \nu(X)), \forall X \in M.$$

In the case $k = 2$ of scalar curvature, they were able to prove the estimate for strictly starshaped 2-convex hypersurfaces. Their proof relies on new test curvature functions and elaborate analytic arguments to overcome the difficulties caused by allowing $f$ to depend on $\nu$.

In this note, we give a simpler proof for the scalar curvature case and we extend the result to space forms $N^{n+1}(K)$, with $K = -1, 0, 1$. Our main result is stated in Theorem 2.1 of section 2 and leads to the existence Theorem 3.3. For related results in the literature see [3], [6], [2] and [8].

2. Prescribed scalar curvature

Let $N^{n+1}(K)$ be a space form of sectional curvature $K = -1, 0,$ and $+1$. Let $g^N := ds^2$ denote the Riemannian metric of $N^{n+1}(K)$. In Euclidean space $\mathbb{R}^{n+1}$, fix the origin $O$ and let $S^n$ denote the unit sphere centered at $O$. Suppose that $(z, \rho)$ are spherical coordinates in $\mathbb{R}^{n+1}$, where $z \in S^n$. The standard metric on $S^n$ induced.

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from $\mathbb{R}^{n+1}$ is denoted by $dz^2$. Let $a$ be constant, $0 < a \leq \infty$, $I = [0, a)$, and $\phi(\rho)$ a positive function on $I$. Then the new metric

\begin{equation}
(2.1) \quad g^N := ds^2 = d\rho^2 + \phi^2(\rho)dz^2.
\end{equation}

on $\mathbb{R}^{n+1}$ is a model of $N^{n+1}$ which is Euclidean space $\mathbb{R}^{n+1}$ if $\phi(\rho) = \rho$, $a = \infty$, the unit sphere $\mathbb{S}^{n+1}$ if $\phi(\rho) = \sin(\rho)$, $a = \pi/2$ and hyperbolic space $\mathbb{H}^{n+1}$ if $\phi(\rho) = \sinh(\rho)$, $a = \infty$.

We recall some formulas for the induced metric, normal, and second fundamental form on $M$ (see [2]). We will denote by $\nabla'$ the covariant derivatives with respect to the standard spherical metric $e_{ij}$, and by $\nabla$ the covariant derivatives with respect to some local orthonormal frame on $M$. Then we have

\begin{equation}
(2.2) \quad g_{ij} = \phi^2 e_{ij} + \rho_i \rho_j, \quad g^{ij} = \frac{1}{\phi^2} \left( e^{ij} - \frac{\rho^i \rho^j}{\phi^2 + |\nabla'\rho|^2} \right).
\end{equation}

\begin{equation}
(2.3) \quad \nu = \frac{(-\nabla' \rho, \phi^2)}{\sqrt{\phi^4 + \phi^2 |\nabla'\rho|^2}},
\end{equation}

and

\begin{equation}
(2.4) \quad h_{ij} = \frac{\phi}{\sqrt{\phi^4 + |\nabla'\rho|^2}} \left( -\nabla'_{ij}\rho + \frac{2\phi'}{\phi} \rho_i \rho_j + \phi \phi' e_{ij} \right).
\end{equation}

Consider the vector field $V = \phi(\rho) \frac{\partial}{\partial \rho}$ in $N^{n+1}(K)$, and define $\Phi(\rho) = \int_0^\rho \phi(r)dr$. Then, $u := \langle V, \nu \rangle$ is the support function. By a straightforward calculation we have the following equations (see [5] lemma 2.2 and lemma 2.6).

\begin{equation}
(2.5) \quad \nabla_{ij} \Phi = \phi' g_{ij} - uh_{ij},
\end{equation}

\begin{equation}
(2.6) \quad \nabla_i u = g^{kl} h_{ik} \nabla_l \Phi,
\end{equation}

and

\begin{equation}
(2.7) \quad \nabla_{ij} u = g^{kl} \nabla_k h_{ij} \nabla_l \Phi + \phi' h_{ij} - u g^{kl} h_{ik} h_{jl}.
\end{equation}

Now let $\Gamma_k$ be the connected component of $\{ \lambda \in \mathbb{R}^n : \sigma_k(\lambda) > 0 \}$, where

$$
\sigma_k = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}
$$

is the k-th mean curvature. $M := \{(z, \rho(z)) : z \in \mathbb{S}^n\}$ is an embedded hypersurface in $N^{n+1}$. We call $\rho$ k-admissible if the principal curvatures $(\lambda_1(\rho(z)), \ldots, \lambda_n(\rho(z)))$ of
\(\mathcal{M}\) belong to \(\Gamma_k\). Our problem is to study a smooth positive 2-admissible function \(\rho\) on \(S^n\) satisfying

\[
\sigma_2(\lambda(b)) = \psi(V, \nu),
\]

where \(b = \{b_{ij}\} = \{\gamma^k h_{kl} \gamma^l\}, \{h_{ij}\}\) is the second fundamental form of \(\mathcal{M}\), and \(\gamma^{ij}\) is \(\sqrt{g^{-1}}\). Equivalently, we study the solution of the following equation

\[
F(b) = \left(\frac{n}{2}\right)^{(-1/2)} \sigma_2(\lambda(b))^{1/2} = f(\lambda(b_{ij})) = \overline{\psi}(V, \nu).
\]

Now we are ready to state and prove our main result.

**Theorem 2.1.** Suppose \(\mathcal{M} = \{(z, \rho(z)) \mid z \in S^n\} \subset N^{n+1}\) is a closed 2-convex hypersurface which is strictly starshaped with respect to the origin and satisfies equation (2.9) for some positive function \(\overline{\psi}(V, \nu) \in C^2(\Gamma)\), where \(\Gamma\) is an open neighborhood of the unit normal bundle of \(\mathcal{M}\) in \(N^{n+1} \times S^n\). Suppose also we have uniform control 0 \(< R_1 \leq \rho(z) \leq R_2 < a, |\rho|_{C^1} \leq R_3\). Then there is a constant \(C\) depending only on \(n, R_1, R_2, R_3\) and \(|\overline{\psi}|_{C^2}\), such that

\[
(2.10) \max_{z \in S^n} |\kappa_i(z)| \leq C.
\]

**Proof.** Since \(\sigma_1(\kappa) > 0\) on \(\mathcal{M}\), it suffices to estimate from above, the largest principal curvature of \(\mathcal{M}\). Consider

\[
M_0 = \max_{x \in \mathcal{M}} e^{\beta \phi} \frac{\kappa_{\text{max}}}{u - a},
\]

where \(u \geq 2a\) and \(\beta\) is a large constant to be chosen (we will always assume \(\beta \phi + K > 0\)). Then \(M_0\) is achieved at \(x_0 = (z_0, \rho(z_0))\) and we may choose a local orthonormal frame \(e_1, \ldots, e_n\) around \(x_0\) such that \(h_{ij}(x_0) = \kappa_i \delta_{ij}\), where \(\kappa_1, \ldots, \kappa_n\) are the principal curvatures of \(\Sigma\) at \(x_0\). We may assume \(\kappa_1 = \kappa_{\text{max}}(x_0)\). Thus at \(x_0\), \(\log h_{11} - \log (u - a) + \beta \Phi\) has a local maximum. Therefore,

\[
0 = \frac{\nabla_i h_{11}}{h_{11}} - \frac{\nabla_i u}{u - a} + \beta \Phi_i,
\]

and

\[
0 \geq \frac{\nabla_i h_{11}}{h_{11}} - \left(\frac{\nabla_i h_{11}}{h_{11}}\right)^2 - \frac{\nabla_i u}{u - a} + \left(\frac{\nabla_i u}{u - a}\right)^2 + \beta \Phi_{ii}.
\]

By the Gauss and Codazzi equations, we have \(\nabla_k h_{ij} = \nabla_j h_{ik}\) and

\[
(2.13) \nabla_{11} h_{ii} = \nabla_{ii} h_{11} + h_{11} h_{ii}^2 - h_{ii}^2 h_{11} + K(h_{11} \delta_{i1} \delta_{i1} - h_{i1} \delta_{ii} + h_{ii} - h_{i1} \delta_{i1}).
\]
Therefore,
(2.14) \[
F_{ii}^i \nabla_{ii} h_{ii} = F_{ii}^i \nabla_{ii} h_{11} + \kappa_1 \sum_i f_i \kappa_i^2 - \kappa_1^2 \sum_i f_i \kappa_i + K \left( -\kappa_1 \sum_i f_i + \sum f_i \kappa_i \right)
= \sum_i f_i \nabla_{ii} h_{11} + \kappa_1 \sum_i f_i \kappa_i^2 - \overline{\psi} \kappa_1^2 + K \left( -\kappa_1 \sum_i f_i + \overline{\psi} \right)
\]

Covariantly differentiating equation (2.9) twice yields
(2.15) \[
F_{ii}^i h_{ii1} = \overline{\psi}_V(\nabla \epsilon V) + h_{ks} \overline{\psi}_V(e_s)
\]
so that
(2.16) \[
\left| \sum_i f_i h_{ii1} \Phi_s \right| \leq C(1 + \kappa_1)
\]
and
(2.17) \[
F_{ii}^i h_{ii1} + F_{ij}^{ij,kl} h_{11} h_{kl} = \nabla_{11}(\overline{\psi}) \geq -C(1 + \kappa_1^2) + h_{11s} \overline{\psi}_V(e_s)
\]

Combining (2.17) and (2.14) and using (2.11), (2.12) gives
\[
0 \geq \frac{1}{\kappa_1} \left\{ -C(1 + \kappa_1^2 + \beta \kappa_1) - F_{ij}^{ij,kl} \nabla_{1j} h_{ij} \nabla_{1k} h_{kl} - \kappa_1 \sum_i f_i \kappa_i^2 + \kappa_1^2 \overline{\psi} - K \left( -\kappa_1 \sum_i f_i + \overline{\psi} \right) \right\}
- \frac{1}{\kappa_1^2} \sum_i f_i |\nabla_{ii} h_{11}|^2 - \frac{1}{u-a} \sum_i f_i \left\{ h_{ii1} \Phi_s - u \kappa_i^2 + \phi' \kappa_i \right\} + \sum_i f_i \frac{|\nabla_i u|^2}{(u-a)^2} - u \beta \overline{\psi} + \beta \phi' \sum_i f_i
\geq -C(\kappa_1 + \beta) - \frac{1}{\kappa_1} F_{ij}^{ij,kl} \nabla_{1j} h_{ij} \nabla_{1k} h_{kl} + \frac{a}{u-a} \sum_i f_i \kappa_i^2 + (\beta \phi' + K) \sum_i f_i
- \frac{1}{\kappa_1^2} \sum_i f_i |\nabla_{ii} h_{11}|^2 + \sum_i f_i \frac{|\nabla_i u|^2}{(u-a)^2}
\]
In other words,
(2.18) \[
0 \geq -C(\kappa_1 + \beta) - \frac{1}{\kappa_1} F_{ij}^{ij,kl} \nabla_{1j} h_{ij} \nabla_{1k} h_{kl} + \frac{a}{u-a} \sum_i f_i \kappa_i^2
+ (\beta \phi' + K) \sum_i f_i - \frac{1}{\kappa_1^2} \sum_i f_i |\nabla_{ii} h_{11}|^2 + \sum_i f_i \frac{|\nabla_i u|^2}{(u-a)^2}.
\]
By (2.11) we have for any $\epsilon > 0$,
(2.19) \[
\frac{1}{\kappa_1^2} \sum_i f_i |\nabla_{ii} h_{11}|^2 \leq (1 + \epsilon^{-1}) \beta^2 \sum_i f_i |\nabla_{ii} \Phi|^2 + \frac{(1 + \epsilon)}{(u-a)^2} \sum_i f_i |\nabla_i u|^2.
\]
Using this in (2.18) we obtain

\begin{equation}
0 \geq -C(\kappa_1 + \beta) - \frac{1}{\kappa_1} F^{ij,kl} \nabla_i h_{ij} \nabla_k h_{kl} + \left( \frac{a}{u - a} - C\epsilon \right) \sum_i f_i \kappa_i^2 \\
+ \left[ \beta \phi' + K - C\beta^2 (1 + \epsilon^{-1}) \right] T ,
\end{equation}

where \( T = \sum_i f_i \). Now we divide the remainder of the proof into two cases.

Case A. Assume \( \kappa_n \leq -\frac{\kappa_1}{n} \). In this case, equation (2.20) implies (here \( \epsilon \) is small controlled multiple of \( a \) and we use \( f_n \geq f_i \) which holds by concavity of \( f \))

\begin{equation}
0 \geq -C(\kappa_1 + \beta) + a \sqrt{\beta} \sum_i f_i \kappa_i^2 - C\beta^2 T \geq -C(\kappa_1 + \beta) + \left( \frac{1}{C} \kappa_1^2 - C\beta^2 \right) T 
\end{equation}

Since \( T \geq 1 \) by the concavity of \( f \), equation (2.21) implies \( \kappa_1 \leq C\beta \) at \( x_0 \).

Case B. Assume \( \kappa_n > -\frac{\kappa_1}{n} \). Let us partition \( \{1, \ldots, n\} \) into 2 parts,

\( I = \{ j : f_j \leq n^2 f_1 \} \) and \( J = \{ j : f_j > n^2 f_1 \} \).

For \( i \in I \), we have (by (2.11)) for any \( \epsilon > 0 \)

\begin{equation}
\frac{1}{\kappa_1^2} f_i |\nabla_i h_{11}|^2 \leq (1 + \epsilon) \sum_i f_i \frac{|\nabla_i u|^2}{(u - a)^2} + C(1 + \epsilon^{-1}) \beta^2 f_1 .
\end{equation}

Inserting this into equation (2.18) gives (for \( \epsilon \) a small controlled multiple of \( a^2 \))

\begin{equation}
0 \geq -C(\kappa_1 + \beta) - \frac{1}{\kappa_1} F^{ij,kl} \nabla_i h_{ij} \nabla_k h_{kl} + \frac{a}{C} \sum_i f_i \kappa_i^2 + (\beta \phi' + K) \sum_i f_i \\
- \frac{1}{\kappa_1^2} \sum_{i \in J} f_i |\nabla_i h_{11}|^2 - C\beta^2 f_1 .
\end{equation}

Now we use an inequality due to Andrews [1] and Gerhardt [4]:

\begin{equation}
- \frac{1}{\kappa_1} F^{ij,kl} \nabla_i h_{ij} \nabla_k h_{kl} \geq \frac{1}{\kappa_1} \sum_{i \neq j} f_i - f_j |\nabla_i h_{11}|^2 \\
\geq \frac{2}{\kappa_1} \sum_{j \geq 2} f_j - f_1 |\nabla_j h_{11}|^2 \\
\geq \frac{2}{\kappa_1^2} \sum_{j \in J} f_j |\nabla_j h_{11}|^2 .
\end{equation}

We now insert (2.24) into (2.23) to obtain

\begin{equation}
0 \geq -C(\kappa_1 + \beta) + \frac{a}{C} \sum f_i \kappa_i^2 + (\beta \phi' + K) \sum f_i - C\beta^2 f_1 .
\end{equation}
Since $\kappa_n > -\frac{1}{n}\kappa_1$ we have that
\[
\sum f_i = \frac{(n-1)\sigma_1}{2^n} > \frac{\kappa_1 - \frac{n-1}{n}\kappa_1}{n^2} = \frac{\kappa_1}{n^2}\psi
\]

We also note that on $\mathcal{M}$, $\phi'$ is bounded below by a positive controlled constant so we may assume $\beta \phi' + K$ is large. Therefore from (2.25) we obtain
\[
(2.26) \quad 0 \geq \left( \frac{\beta \phi' + K}{n^2 \psi} - C \right) \kappa_1 - C \beta + \left( \frac{a}{C_2} \kappa_1^2 - C \beta^2 \right) f_1.
\]

We now fix $\beta$ large enough that $\frac{\beta \phi' + K}{n^2 \psi} > 2C$ which implies a uniform upper bound for $\kappa_1$ at $x_0$. By the definition of $M_0$ we then obtain a uniform upper bound for $\kappa_{\text{max}}$ on $\mathcal{M}$ which implies a uniform upper and lower bound for the principle curvatures. \( \square \)

3. Lower order estimates

In this section, we obtain $C^0$ and $C^1$ estimates for the more general equation:

(3.1) \quad $\sigma_k(\kappa) = \psi(V, \nu)$,

where $k = 1, \cdots, n$.

3.1. $C^0$ estimates. The $C^0$-estimates were proved in [2] but for the reader’s convenience we include the simple proof.

**Lemma 3.1.** Let $1 \leq k \leq n$ and let $\psi \in C^2(N^{n+1} \times S^n)$ be a positive function. Suppose there exist two numbers $R_1$ and $R_2$, $0 < R_1 < R_2 < a$, such that

(3.2) \quad $\psi \left( V, \frac{V}{|V|} \right) \geq \sigma_k(1, \cdots, 1)q^k(\rho), \rho = R_1$,

(3.3) \quad $\psi \left( V, \frac{V}{|V|} \right) \leq \sigma_k(1, \cdots, 1)q^k(\rho), \rho = R_2$,

where $q(\rho) = \frac{1}{\phi} \frac{d\phi}{d\rho}$. Let $\rho \in C^2(S^n)$ be a solution of equation (3.1). Then

$R_1 \leq \rho \leq R_2$. 

Proof. Suppose that \( \max_{z \in \mathbb{S}^n} \rho(z) = \rho(z_0) > R_2 \). Then at \( z_0 \),
\[
g^{ij} = \phi^{-2} e^{ij}, \ h_{ij} = -\nabla_i' \rho + \phi \phi' e_{ij} \geq \phi \phi' e_{ij}, \ b_{ij} \geq q(\rho) \delta_{ij}.
\]
Hence \( \psi(V, \nu)(z_0) = \sigma_k(b_{ij})(z_0) > q^k(R_2) \sigma_k(1, \ldots, 1) \), contradicting (3.3). The proof of (3.2) is similar. \( \square \)

3.2. \( C^1 \) estimates. In this section, we follow the idea of [3] and [6] to derive \( C^1 \) estimates for the height function \( \rho \). In other words, we are looking for a lower bound for the support function \( u \). First, we need the following technical assumption: for any fixed unit vector \( \nu \),

\[
\frac{\partial}{\partial \rho}(\phi(\rho)^k \psi(V, \nu)) \leq 0, \text{ where } |V| = \phi(\rho).
\]

Lemma 3.2. Let \( M \) be a radial graph in \( N^{n+1} \) satisfying (3.1), (3.4) and let \( \rho \) be the height function of \( M \). If \( \rho \) has positive upper and lower bounds, then there is a constant \( C \) depending on the minimum and maximum values of \( \rho \), such that
\[
|\nabla \rho| \leq C.
\]

Proof. Consider \( h = -\log u + \gamma(\Phi(\rho)) \) and suppose \( h \) achieves its maximum at \( z_0 \). We will show that for a suitable choice of \( \gamma(t), u(z_0) = |V(z_0)|, \) that is \( V(z_0) = |V(z_0)| \nu(z_0) \), which implies a uniform lower bound for \( u \) on \( M \). If not, we can choose a local orthonormal frame \( \{e_1, \ldots, e_n\} \) on \( M \) such that \( \langle V, e_1 \rangle \neq 0 \), and \( \langle V, e_i \rangle = 0, i \geq 2 \). Then at \( z_0 \) we have,

\[
h_i = -\frac{u_i}{u} + \gamma' \nabla_i \Phi = 0,
\]

(3.5)

\[
0 \geq h_{ii} = -\frac{u_{ii}}{u} + \left( \frac{u_i}{u} \right)^2 + \gamma' \nabla_{ii} \Phi + \gamma''(\nabla_i \Phi)^2
\]

(3.6)

\[
= -\frac{1}{u} \left( h_{i11} \nabla_1 \Phi + \phi' h_{ii} - uh_{ii}^2 \right) + [(\gamma')^2 + \gamma''](\nabla_i \Phi)^2 + \gamma'(\phi' g_{ii} - h_{ii} u). \]

Equation (3.5) gives

(3.7)

\[
h_{11} = u \gamma', \ h_{i1} = 0, \ i \geq 2
\]

so we may rotate \( \{e_2, \ldots, e_n\} \) so that \( h_{ij}(z_0, \rho(z_0)) \) is diagonal. Hence,

(3.8)

\[
0 \geq -\frac{1}{u} \left( \sigma_k^{i1} h_{i1} \nabla_1 \Phi + \phi' k \psi - u \sigma_k^{ii} h_{ii}^2 \right)
\]

\[
+ [(\gamma')^2 + \gamma''](\nabla_1 \Phi)^2 \sigma^{11} + \gamma'(\phi' \sum \sigma_k^{ii} - k \psi u)
\]
Differentiating equation (3.1) with respect to $e_1$ we obtain

\begin{equation}
\sigma_k^{ii} h_{i1} = d_V \psi(\nabla_{e_1} V) + h_{11} d_\nu \psi(e_1).
\end{equation}

Substituting equation (3.9) and (3.7) into (3.8) yields

\begin{equation}
0 \geq -\frac{1}{u} \left[ \langle V, e_1 \rangle d_V \psi(\nabla_{e_1} V) + w\gamma' \langle V, e_1 \rangle d_\nu \psi(e_1) + k\phi' \psi \right] \\
\quad + \sigma_k^{ii} h_{ii}^2 + [(\gamma')^2 + \gamma''] \langle V, e_1 \rangle^2 \sigma_k^{11} + \gamma' \phi' \sigma_k^{ii} - ku\gamma' \psi \\
\quad = -\frac{1}{u} \left[ \langle V, e_1 \rangle d_V \psi(\nabla_{e_1} V) + k\phi' \psi \right] + \sigma_k^{ii} h_{ii}^2 \\
\quad + [(\gamma')^2 + \gamma''] \langle V, e_1 \rangle^2 \sigma_k^{11} + \gamma' \phi' \sum \sigma_k^{ii} - u\gamma' \psi - \gamma' \langle V, e_1 \rangle d_\nu \psi(e_1).
\end{equation}

Our assumption (3.4) is equivalent to

\begin{equation}
k\phi^{k-1} \phi' \psi + \phi^k \frac{\partial}{\partial \rho} \psi(V, \nu) \leq 0,
\end{equation}
or

\begin{equation}
k\phi' \psi + d_V \psi(V, \nu) \leq 0.
\end{equation}

Since at $z_0$, $V = \langle V, e_1 \rangle e_1 + \langle V, \nu \rangle \nu$

\begin{equation}
d_V \psi(V, \nu) = \langle V, e_1 \rangle d_V \psi(\nabla_{e_1} V) + \langle V, \nu \rangle d_V \psi(\nabla_\nu V).
\end{equation}

Therefore,

\begin{equation}
0 \geq \sigma_k^{ii} h_{ii}^2 + [(\gamma')^2 + \gamma''] \langle V, e_1 \rangle^2 \sigma_k^{11} + \gamma' \phi' \sum \sigma_k^{ii} \\
\quad - u\gamma' \psi - \gamma' \langle V, e_1 \rangle d_\nu \psi(e_1) + d_V \psi(\nabla_\nu V)
\end{equation}

Now let $\gamma(t) = \frac{\alpha}{t}$, where $\alpha > 0$ is sufficiently large. Since $h_{11} \leq 0$ at $z_0$, and

\begin{equation}
\sum \sigma_k^{ii} = (n - k + 1)\sigma_{k-1},
\end{equation}

we have that

\begin{equation}
\sigma_k^{11} = \sigma_{k-1}(\kappa_{k_1}) \geq \sigma_{k-1} \geq \sigma_k^{1-k} = \psi^{\frac{k-1}{k}}.
\end{equation}

Therefore

\begin{equation}
[(\gamma')^2 + \gamma''] \langle V, e_1 \rangle^2 \sigma_k^{11} + \sigma_k^{ii} h_{ii}^2 + \gamma' \phi' \sum \sigma_k^{ii} \geq C\alpha^2 \sigma_k^{11},
\end{equation}

for some $C$ depending on $|\rho|_{C^0}$.

We conclude that

\begin{equation}
0 \geq C\alpha^2 \psi^{\frac{k-1}{k}} - \alpha |V| |d_\nu \psi(e_1)| - |d_V \psi(\nabla_\nu V)|,
\end{equation}

which leads to a contradiction when $\alpha$ is large. Therefore at $z_0$ we have $u = |V|$, which completes the proof. \qed
By a standard continuity argument (see [3]), we can prove the following theorem.

**Theorem 3.3.** Suppose \( \psi \in C^2(\bar{B}_{r_2} \setminus B_{r_1} \times S^n) \) satisfies conditions (3.2), (3.3), and (3.4). Then there exists a unique \( C^{3,\alpha} \) starshaped solution \( M \) satisfying equation (2.8).

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