9.3.3 For part a), the solutions to the eigenvalue problem take the form \( \phi_n(x) = \sin(\sqrt{\lambda_n}x) \), \( \lambda_n = \left( \frac{n}{L} \right)^2 (n - \frac{1}{2})^2 \). Hence the method of eigenfunction expansion gives that the solution to the nonhomogeneous problem is

\[
u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin(\sqrt{\lambda_n}x)\]

with

\[
a_n(t) = e^{-k\lambda_n t}a_n(0) + \frac{2}{L} \int_0^t \int_0^L e^{-\lambda_n k(t-t_0)} \sin(\sqrt{\lambda_n}x_0) Q(x_0, t_0) dx_0 dt_0
\]

\[
a_n(0) = \frac{2}{L} \int g(x_0) \sin(\sqrt{\lambda_n}x_0) dx_0.
\]

Collecting all of this gives the answer to part b), the Green’s function \( \tilde{G}(x, t; x_0, t_0) \) for the time dependent problem

\[
\tilde{G}(x, t; x_0, t_0) = \frac{2}{L} \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n}x_0) \sin(\sqrt{\lambda_n}x) e^{-\lambda_n k(t-t_0)}.
\]

Finally, by following the approach in 9.3.1, we get the following relationship between the Green’s function for the time dependent problem \( \tilde{G}(x, t; x_0, t_0) \) and the Green’s function for the steady state problem

\[
\frac{d^2u}{dx^2} = f
\]

where the second inequality follows after considering boundary conditions \( G(0, x_0) = 0 \), \( \frac{dG}{dx_0}(L, x) = 0 \). Next consider continuity and jump conditions at \( x_0 \) to obtain

\[
0 = G(x_0+, x_0) - G(x_0-, x_0) = d - ax_0
\]

9.3.5/9.3.6 The main point of these two problems is to develop the following Green’s function for the problem

\[
G(x, x_0) = \begin{cases} 
-x & x < x_0 \\
-x_0 & x > x_0
\end{cases}
\]

The main point of these problems is that there are a few ways to derive this.

**Direct Method** The direct method begins with the observation that when \( x \neq x_0 \), \( \frac{d^2G}{dx^2}(x, x_0) = 0 \) and hence \( G \) is of the form

\[
G(x, x_0) = \begin{cases} 
a x + b & x < x_0 \\
ax + d & x > x_0
\end{cases} = \begin{cases} 
a x & x < x_0 \\
d & x > x_0
\end{cases}
\]

where the second inequality follows after considering boundary conditions \( G(0, x_0) = 0 \), \( \frac{dG}{dx_0}(L, x) = 0 \). Next consider continuity and jump conditions at \( x_0 \) to obtain

\[
0 = G(x_0+, x_0) - G(x_0-, x_0) = d - ax_0
\]
\[ 1 = \frac{dG}{dx}(x_0^+, x_0) - \frac{dG}{dx}(x_0^-, x_0) = 0 - a \]

so that \( a = -1, d = -x_0 \), which is the desired solution.

**Direct Integration** Direct integration yields the formula (9.3.17) in Haberman, where a double integral is taken over the region \( \{(x_0, x) : 0 \leq x \leq x_0 \leq \bar{x}\} \). Care is required when changing order of integration in the formula (you have to use a bit of vector calculus to get it)

\[
u(x) = \int_0^x \int_0^x f(\bar{x}) \, d\bar{x} \, dx_0 + c_1 x + c_2 = \int_0^x f(\bar{x}) \left( \int_0^x \, dx_0 \right) \, d\bar{x} + c_1 x + c_2
\]

\[
= \int_0^x (x - \bar{x}) f(\bar{x}) \, d\bar{x} + c_1 x + c_2.
\]

Considering BC’s now gives \( c_2 = 0 \) and \( c_1 \) determined by

\[
0 = \frac{du}{dx}(L) = \int_0^L f(\bar{x}) \, d\bar{x} + c_1
\]

plugging this back into the formula for \( u(x) \) now yields the formula

\[
u(x) = -x \int_0^L f(\bar{x}) \, d\bar{x} - \int_0^x \bar{x} f(\bar{x}) \, d\bar{x} = \int_0^L G(x, \bar{x}) f(\bar{x}) \, d\bar{x}.
\]

**Variation of Parameters** The easiest thing to do here is to consider homogeneous solutions \( u_1(x) = x \) and \( u_2(x) = 1 \) so that \( u_1(0) = 0 \) and \( u_2(0) = 0 \). The Wronskian here is \( W = -1 \) and hence

\[
\frac{dv_1}{dx} = -\frac{fu_2}{W} = f(x) \quad \frac{dv_2}{dx} = \frac{fu_1}{W} = -x f(x)
\]

Since \( v_1, v_2 \) are chosen so that \( u' = v_1 u_1' + v_2 u_2' \) we have that

\[
0 = \frac{du}{dx}(L) = v_1(L) \frac{du_1}{dx}(L) + v_2(L) \frac{du_2}{dx}(L) = L v_1(L)
\]

\[
0 = u(0) = u_1(0)v_1(0) + u_2(0)v_2(0) = v_2(0)
\]

Thus by taking antiderivatives, it can be seen that

\[
v_1(x) = -\int_0^L f(x_0) \, dx_0 \quad v_2(x) = \int_0^x -x_0 f(x_0) \, dx_0
\]

and hence

\[
u(x) = -x \int_0^L f(x_0) \, dx_0 - \int_0^x x_0 f(x_0) \, dx_0 = \int_0^L G(x, x_0) f(x_0) \, dx_0.
\]

**9.5.11** It’s easiest to start by searching for a Green’s function of the form

\[
G(x, x_0) = \begin{cases} a \sin x + b \cos x & x < x_0 \\ c \sin(x - L) + d \cos(x - L) & x > x_0 \end{cases}
\]

This is because considering boundary conditions \( G(x, 0) = G(x, L) = 0 \) gives us the nice simplification

\[
G(x, x_0) = \begin{cases} a \sin x & x < x_0 \\ c \sin(x - L) & x > x_0 \end{cases}
\]
To solve for $a$, $c$ we now need to consider behavior at $x_0$, namely $G(x_0+,x_0) = G(x_0-,x_0)$, $\frac{dG}{dx}(x_0+,x_0) - \frac{dG}{dx}(x_0-,x_0) = 1$. Computing $\frac{dG}{dx}$ it can be seen that $a$, $c$ must satisfy the matrix equation

$$\begin{bmatrix} \sin x_0 & -\sin(x_0 - L) \\ \cos x_0 & -\cos(x_0 - L) \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In order for the matrix equation to be solvable, the determinant of the $2 \times 2$ matrix on the left must be nonzero. Using angle formulas, the determinant equals

$\sin x_0 \cos(x_0 - L) - \cos x_0 \sin(x_0 - L) = \sin x_0[\cos x_0 \cos(-L) - \sin x_0 \sin(-L)] - \cos x_0[\sin x_0 \cos(-L) + \sin(-L) \cos x_0] = -\sin^2 x_0 \sin(-L) - \cos^2 x_0 \sin(-L) = \sin L$

which is nonzero when $L \neq n\pi$. By inverting the matrix it can now be seen that $a = \sin(x_0 - L)/\sin L$, $c = \sin x_0/\sin L$ so that the Green’s function becomes

$$G(x,x_0) = \begin{cases} \frac{\sin(x-x_0)\sin x}{\sin L} & x < x_0 \\ \frac{\sin(x-x_0)\sin x_0}{\sin L} & x > x_0 \end{cases}$$

which is easily checked to satisfy Maxwell reciprocity.

9.5.12 Part a) takes some care as formula (9.3.51) in Haberman does not hold due to the insulated left end condition that $G(x,x_0)$ satisfies. Begin with Green’s identity in $x_0$ applied to $u(x_0)$ and $v(x_0) = G(x,x_0)$

$$\int_0^L u(x_0) \frac{d^2 G}{dx_0^2} - G(x,x_0) \frac{d^2 u}{dx_0^2} \ dx_0 = u(x_0) \frac{dG(x,x_0)}{dx_0} \bigg|_{x_0=L} - G(x,x_0) \frac{du}{dx_0} \bigg|_{x_0=0}.$$

For each fixed $x$, $G(x,0) = 0$, $\frac{dG(x,0)}{dx_0} = 0$. Hence, simplifying both sides of the identity gives

$$\int_0^L u(x_0)\delta(x-x_0) - G(x,x_0)f(x_0) \ dx_0 = -u(0)\frac{dG(x,0)}{dx_0} - G(x,L)\frac{du}{dx_0}(L).$$

Finally, substituting in the boundary conditions for $u$ yields

$$u(x) - \int_0^L G(x,x_0)f(x_0) \ dx_0 = -A\frac{dG(x,0)}{dx_0} - G(x,L)B$$

and thus after computing $\frac{dG(x,0)}{dx_0} = -1$ and observing $G(x,L) = -x$

$$u(x) = \int_0^L G(x,x_0)f(x_0)dx_0 + A + Bx.$$ 

For part b), formula (9.3.51) in Haberman holds since the Green’s function is zero at both ends. Hence

$$u(x) = \int_0^L G(x,x_0)f(x_0)dx_0 + \frac{B\sin x}{\sin L} - \frac{A\sin(x-L)}{\sin L}.$$