THE STABLE SYMPLECTIC CATEGORY AND GEOMETRIC QUANTIZATION

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ABSTRACT. We study a stabilization of the symplectic category introduced by A. Weinstein as a domain for the geometric quantization functor. The symplectic category is a topological category with objects given by symplectic manifolds, and morphisms being suitable lagrangian correspondences. The main drawback of Weinstein’s symplectic category is that composition of morphisms cannot always be defined. Our stabilization procedure rectifies this problem while remaining faithful to the original notion of composition. The stable symplectic category is enriched over the category of spectra (in particular, its morphisms can be described as infinite loop spaces of stable lagrangians), and it possesses several appealing properties that are relevant in the context of geometric quantization.

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1. INTRODUCTION

Motivated by earlier work by Guillemin and Sternberg [8], A. Weinstein [10, 12] introduced the symplectic category as a domain category for constructing the (yet to be completely defined) geometric quantization functor. The objects of this topological category are symplectic manifolds, and morphisms between two symplectic manifolds \((M, \omega)\) and \((N, \eta)\) are defined as lagrangian correspondences inside \(\mathcal{M} \times \mathcal{N}\), where the conjugate symplectic manifold \(\mathcal{M}\) is defined by the pair \((M, -\omega)\). Geometric quantization is an attempt at constructing a canonical representation of this category. There are many categories closely related to the symplectic category (for example the Fukaya 2-category), that we shall not consider in this document.

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Composition of two lagrangians $L_1 \subset M \times N$ and $L_2 \subset N \times K$ in the symplectic category is defined as $L_1 \ast L_2 \subset \overline{M} \times K$ given by the image of the projection

$$\pi_{1,4} : \overline{M} \times N \times \overline{N} \times K \longrightarrow \overline{M} \times K,$$

restricted to the intersection of the two subspaces: $L_1 \times L_2$ and $\overline{M} \times \Delta(N) \times K$, where $\Delta(N) \subset N \times \overline{N}$ is the diagonal submanifold. The first observation to make is that the symplectic category is not strictly a category because composition as defined above does not always yield a lagrangian submanifold. For composition to yield a lagrangian submanifold, the intersection that is used to define it must be transverse. One way to fix this problem was introduced by Wehrheim and Woodward [13] where they consider the free category generated by all the correspondences modulo the obvious relation if the pair of composable morphisms is transverse.

In this document, we introduce another method of extending the symplectic category into an honest category. We introduce a moduli space of stable lagrangian immersions in a symplectic manifold of the form $\overline{M} \times N$. This moduli space can be described as the infinite loop space corresponding to a certain Thom spectrum. Taking this as the space of morphisms defines the stable symplectic category that is naturally enriched over the monoidal category of spectra (under smash product). Composition in this stable symplectic category remains faithful to the original definition introduced by Weinstein. In addition, it factors through the equivalence in the symplectic category that identifies two symplectic manifolds $M$ and $N$ if $M \times \mathbb{C}^k$ becomes equivalent to $N \times \mathbb{C}^k$ for some $k$.

Having stabilized, we notice the appearance of structure relating to the categorical aspects of geometric quantization. To begin with, we observe that the morphism spectra in our category are conjugate dual (see 4.3). In particular, there is an intersection pairing between stable lagrangians inside a symplectic manifold and those inside its conjugate. Endomorphisms of an object $(M, \omega)$ in our category can be seen as a homotopical notion of the “algebra of observables”. Our framework allows us to construct canonical stable representations of this algebra (see 4.5, and 6.7). Extending coefficients on our category by a flat line bundle, we also recover a potential receptacle for the symbols of integral operators in geometric quantization (see section 6). In addition, a derived version of geometric quantization can be realized as the 2-trace of this symbol (see 6.10).

Before we begin, let us say a word about our notation. In the following, given a pair of symplectic manifolds $M, N$, we shall construct two canonical bundles associated to the pair, which we shall denote by the letters $\tau$ and $\zeta$. For the sake of simplicity, we will use the same letters for different pairs of manifolds. The reader should interpret those bundles in the context of the pair being considered.

We begin by thanking Gustavo Granja for his interest in this project and his hospitality at the IST (Lisbon) where this project started. We thank Jack Morava for his continued interest and encouragement, and for sharing [9] with us. We would like to also thank David Ayala, John Klein and John Lind for helpful conversations related to various parts of this project. And finally, we would like to acknowledge our debt to Peter Landweber for carefully reading an earlier version of this manuscript and providing several very helpful suggestions.
2. Stable Lagrangian Immersions and Their Moduli Space

In this section we will describe the stabilization procedure that we will apply to the symplectic category in later sections. To begin with consider a symplectic manifold \((M^{2m}, \omega)\) endowed with a compatible almost complex structure. Let \(\tau : M \to BU(m)\) denote the map that classifies its (complex) tangent bundle. We will assume that the cohomology class of \(\omega\) is a multiple of the first Chern class of \(M\). Now fix the model for \(BU(m)\) as the space of \(m\)-dimensional complex planes in \(\mathbb{C}^\infty\). In particular, the universal space \(EU(m)\) can be identified with the space of all orthonormal \(m\)-frames in \(\mathbb{C}^\infty\). We will choose \(EU(m)/O(m)\) as our model for \(BO(m)\). So we have a bundle \(BO(m) \to BU(m)\) with fiber \(U(m)/O(m)\). Consider the pullback diagram:

\[
\begin{array}{ccc}
S(M) & \xrightarrow{\zeta} & BO(m) \\
\downarrow & & \downarrow \\
M & \xrightarrow{\tau} & BU(m)
\end{array}
\]

**Definition 2.1.** Now let \(X^m\) be an arbitrary \(m\)-manifold, and let \(\zeta\) be an \(m\)-dimensional real vector bundle over a space \(B\). By a \(\zeta\)-structure on \(X\) we shall mean a bundle map \(\tau(X) \to \zeta\), where \(\tau(X)\) is the tangent bundle of \(X\).

**Claim 2.2.** The space of lagrangian immersions of \(X\) into \(M\) is equivalent to the space of \(\zeta\)-structures on the tangent bundle of \(X\), where \(\zeta\) is the vector bundle over \(S(M)\) defined by the above pullback.

**Proof.** A \(\zeta\)-structure on \(\tau(X)\) is the same thing as a map of \(X\) to \(M\), along with an inclusion of \(\tau(X)\) as an orthogonal (lagrangian) sub-bundle inside the pullback of the tangent bundle of \(M\). Now the pullback of \([\omega]\) to \(H^2(X, \mathbb{R})\) factors through \(H^2(BO(m), \mathbb{R})\), by our assumption that \([\omega] = c_1\) up to a scalar. Since \(H^2(BO(m), \mathbb{R}) = 0\), the h-principle [6] may now be invoked to show that the space of maps described above is equivalent to the space of lagrangian immersions of \(X\) in \(M\).

Motivated by [7], consider the Thom spectrum \(S(M)^{-\zeta}\). In a moment we will construct a family of such spectra and identify the stable spectrum given by their colimit. Geometrically, one may also describe a stabilization procedure on the moduli space of immersed lagrangians in \(M\), and show how the infinite loop space for the former agrees with the stabilized moduli space (see the discussion at the end of the section for a precise statement). Let us provisionally observe that for a compact manifold \(X\) admitting a lagrangian immersion into \(M\), an explicit stable map \([X] : S^0 \to S(M)^{-\zeta}\) is constructed as the composite:

\[
[X] : S^0 \to X^{-\tau(X)} \to S(M)^{-\zeta}
\]

where the first map is the Pontrjagin–Thom construction on some embedding of \(X \subset \mathbb{R}^\infty\), and the next map is given by the (negative of the) \(\zeta\)-structure on \(X\).

**Remark 2.3.** Notice that the group of reparametrizations of the stable tangent bundle of \(X\) acts on \(X^{-\tau(X)}\), and may potentially change the immersion class of \(X\). We thank Thomas Kargh for pointing this out.
Now if \((M, \omega)\) and \((N, \eta)\) are two symplectic manifolds, the above construction yields a canonical map that represents the cartesian product of immersions:

\[
\mu : S(M)^{-\zeta_M} \land S(N)^{-\zeta_N} \to S(M \times N)^{-\zeta_{M \times N}}
\]

Taking \((N, \eta)\) to be \(\mathbb{C}\) with the standard symplectic structure, and including \(S^{-1} \to S(\mathbb{C})^{-\zeta}\) as the Thom spectrum over the point in \(S(\mathbb{C})\) represented by the line \(\mathbb{R} \subset \mathbb{C}\) we get:

\[
\Sigma^{-1} S(M)^{-\zeta_M} \to S(M \times \mathbb{C})^{-\zeta_{M \times \mathbb{C}}}
\]

We may suspend this map to get a map:

\[
S(M)^{-\zeta_M} \to \Sigma S(M \times \mathbb{C})^{-\zeta_{M \times \mathbb{C}}}
\]

**Definition 2.4.** Define the Thom spectrum representing the space of stable lagrangian immersions in \(M\) to be the colimit:

\[
S(M)^{-\zeta} := \hocolim_k \Sigma^k S(M \times \mathbb{C}^k)^{-\zeta_k}
\]

where \(\zeta_k\) denotes \(\zeta_{M \times \mathbb{C}^k}\). Alternatively, this spectrum can be seen as the Thom spectrum for the stable bundle \(\zeta\) defined using the pullback:

\[
\begin{array}{ccc}
S(M) & \xrightarrow{\zeta} & \mathbb{Z} \times BO \\
\downarrow & & \downarrow \\
M & \xrightarrow{\tau} & \mathbb{Z} \times BU
\end{array}
\]

where \(\tau\) is the stable tangent bundle of \(M\) of virtual (complex) dimension \(m\).

**Remark 2.5.** Notice that by construction, we have a canonical equivalence:

\[
S(M \times \mathbb{C})^{-\zeta} = \Sigma^{-1} S(M)^{-\zeta}
\]

where we have abused notation and used the same letter \(\zeta\) to denote the stable bundle over \(S(M)\), and over \(S(M \times \mathbb{C})\). We take this opportunity to introduce an abusive notation of using the same letter \(\zeta\) (regardless of the manifold) to mean the bundle defined by the above pullback. Hopefully the bundle will be clear from the context of the manifold, and this notation will cause no confusion.

The product map stabilizes to a map:

\[
\mu : S(M)^{-\zeta} \land S(N)^{-\zeta} \to S(M \times N)^{-\zeta}
\]

In the case of \(M = \ast\), the spectrum \(S(\ast)^{-\zeta}\) is the Thom spectrum of the (formal negative of the) stable vector bundle over \((U/O)\) of virtual dimension zero given by the inclusion \((U/O) \to BO\). In particular, \(S(\ast)^{-\zeta}\) is a commutative ring spectrum, and the product map:

\[
\mu : S(\ast)^{-\zeta} \land S(N)^{-\zeta} \to S(N)^{-\zeta}
\]

describes \(S(N)^{-\zeta}\) as a module over \(S(\ast)^{-\zeta}\).

**Example 2.6.** Consider the example of a cotangent bundle \(M = T^*X\), on a smooth \(m\)-dimensional manifold \(X\) and endowed with the canonical symplectic form. Now since the tangent bundle \(\tau\) of \(M\) admits a lift to \(\mathbb{Z} \times BO\), the space \(S(T^*X)\) is homotopy equivalent to \((U/O) \times X\). In particular, stably, a compact lagrangian immersion \(L \to T^*X\) is represented by a \(\zeta \times \tau(X)\) structure on \(L\), where \(\zeta\) is the virtual zero dimensional bundle over \((U/O)\) defined above. Hence the stable nature of caustics (defined as the critical set of the projection map \(\pi : L \to X\)) is measured by a “universal Maslov structure” on \(L\),
which we define as a bundle map from the stable fiberwise tangent bundle of $\pi$, to the bundle $\zeta$, that lifts the universal Maslov class $L \to U/O$. Now it is easy to see that:

$$\mathbb{S}(T^*X)^{-\zeta} = \mathbb{S}(\ast)^{-\zeta} \wedge X^{-\tau(X)}$$

where $X^{-\tau(X)}$ denotes the Thom spectrum of the formal negative of $\tau(X)$. If $X$ were compact, then $\mathbb{S}(\ast)^{-\zeta} \wedge X^{-\tau(X)}$ is equivalent to $\text{Map}(X, \mathbb{S}(\ast)^{-\zeta})$. This can be interpreted as saying that a stable lagrangian $L$ in $T^*X$ can be interpreted as a family of virtual dimension zero stable lagrangians parametrized as fibers of the map $\pi : L \to X$.

The geometric meaning of the moduli space of stable lagrangians:

Let us briefly describe the geometric objects that our moduli space of stable immersions captures. Fix $k > 0$, and consider the Thom spectrum $\Sigma^k \mathbb{S}(M \times \mathbb{C}^k)^{-\zeta_k}$, where the notation is borrowed from the earlier part of this section. Using methods described in [3] (Sec. 4.4), one can interpret the infinite loop space $\Omega^\infty(-k)(\mathbb{S}(M \times \mathbb{C})^{-\zeta})$ as the moduli space of manifolds $W^{m+k} \subset \mathbb{R}^\infty \times \mathbb{R}^k$, with a proper projection onto $\mathbb{R}^k$, and endowed with a lagrangian immersion $W^{m+k} \to M \times \mathbb{C}^k$. In this interpretation, the stable map:

$$\Sigma^k \mathbb{S}(M \times \mathbb{C}^k)^{-\zeta_k} \to \Sigma^{k+1} \mathbb{S}(M \times \mathbb{C}^{k+1})^{-\zeta_{k+1}}$$

destabilizes to the map that simply sends $W^{m+k}$ to $W^{m+k} \times \mathbb{R}$. Notice that the “virtual dimension” $m$ of the stable lagrangian is well defined. We thank David Ayala for patiently helping us understand this point of view.

Notice that for an (ordinary) lagrangian immersion $L \to M$ to represent a point in the moduli space $\Omega^\infty(\mathbb{S}(M)^{-\zeta})$, the manifold $L$ must be compact. Assume now that $L$ is a (possibly non-compact) manifold immersing into $M$. Then one may still construct a point in the moduli space, provided one has a 1-form $\alpha$ on $L \times \mathbb{R}^k$ for some $k$, with the property that $\alpha^* \iota_\theta : L \times \mathbb{R}^k \to \mathbb{R}^k$ is proper. Here $\iota_\theta$ is the linear projection $T^*(L \times \mathbb{R}^k) \to (\mathbb{R}^k)^* = \mathbb{R}^k$, which is being pulled back to $L \times \mathbb{R}^k$ along $\alpha$. In particular, any function $\phi$ on $L \times \mathbb{R}^k$ so that $\alpha = d\phi$ is proper over $\mathbb{R}^k$ gives rise to a point in the stable moduli space. Such functions are a natural analog of the theory of phase functions in our context [12].

3. The Stable Symplectic Category

Let us now describe the Stable Symplectic category $\mathbb{S}$. By definition, the objects of this category $\mathbb{S}$, will be symplectic manifolds $(M, \omega)$ (see remark 3.5). We will further assume that we have endowed each symplectic manifold with a compatible almost complex structure. In addition to this we assume for the sake of simplicity that $(M, \omega)$ is monotone, i.e. that the cohomology class of $\omega$ is $c_1(M)$ up to a scalar.\(^1\)

The morphisms in our category $\mathbb{S}$ will naturally have the structure of Thom spectra. Let $(M, \omega)$ and $(N, \eta)$ be two objects. We define the conjugate of $(M, \omega)$ to be the symplectic manifold $\overline{M}$ which has the same underlying manifold as $M$ but with symplectic form $-\omega$.

\(^1\)It is possible to incorporate the case of arbitrary $\omega$ that admit integral representatives up to scalar.
Definition 3.1. The “morphism spectrum” \( \Omega_{\varsigma}(M, N) \) in \( \mathbb{S} \) between \( M \) and \( N \) is defined as: \( \Omega_{\varsigma}(M, N) := \Sigma(M \times N)^{-\varsigma} \). We recall that \( \Sigma(M \times N) \) is defined as the pullback:

\[
\begin{array}{ccc}
\Sigma(M \times N) & \xrightarrow{\varsigma} & Z \times BO \\
\downarrow & & \downarrow \\
M \times N & \xrightarrow{\tau} & Z \times BU
\end{array}
\]

Assume now that \( M = N \) is a compact symplectic manifold. Then the diagonal lagrangian embedding \( \Delta : N \to N \times N \) induces a canonical isomorphism between \( TN \otimes \mathbb{C} \) and the restriction of the tangent bundle of \( N \times N \) along \( \Delta \). This defines a stable map \([id] : S^0 \to \Omega_{\varsigma}(N, N)\). Once we define composition, we shall see that \([id]\) is indeed the identity in the category \( \mathbb{S} \).

The next step is to define composition. Consider \( k + 1 \) objects \( M_i \) with \( 0 \leq i \leq k \), and let the space \( \Sigma(M_{0, \ldots, k}) \) be defined by the pullback:

\[
\begin{array}{ccc}
\Sigma(M_{0, \ldots, k}) & \xrightarrow{\xi_i} & \Sigma(M_0 \times M_1) \times \cdots \times \Sigma(M_{k-1} \times M_k) \\
\downarrow & & \downarrow \\
M_0 \times \Delta(M_i)^{k-1} \times M_k & \xrightarrow{\Delta^{k-1}} & M_0 \times (M_i \times M_i)^{k-1} \times M_k
\end{array}
\]

Let \( \varsigma_i \) denote the individual structure maps \( \Sigma(M_{i-1} \times M_i) \to Z \times BO \), and let \( \eta(\Delta^{k-1}) \) denote the normal bundle of \( \Delta^{k-1} \). Performing the Pontrjagin–Thom construction along the top horizontal map in the above diagram yields a morphism of spectra:

\[
\Omega_{\varsigma_1}(M_0, M_1) \wedge \cdots \wedge \Omega_{\varsigma_k}(M_{k-1}, M_k) \to \Sigma(M_{0, \ldots, k})^{-\lambda_i}
\]

where \( \lambda_i : \Sigma(M_{0, \ldots, k}) \to Z \times BO \) is the formal difference of the bundle \( \bigoplus \varsigma_i \) and the pullback bundle \( \xi_i^*(\eta(\Delta^{k-1})) \).

Now let \( \pi : \Sigma(M_{0, \ldots, k}) \to M_0 \times M_k \) denote the projection map onto the first and last factor. Then one checks that there is a canonical commutative diagram:

\[
\begin{array}{ccc}
\Sigma(M_{0, \ldots, k}) & \xrightarrow{\lambda_i} & Z \times BO \\
\downarrow & & \downarrow \\
M_0 \times M_k & \xrightarrow{\tau} & Z \times BU
\end{array}
\]

By the universal definition of \( \Sigma(M_0 \times M_k) \), this induces a canonical map from \( \Sigma(M_{0, \ldots, k}) \) to \( \Sigma(M_0 \times M_k) \) that pulls \( \varsigma \) back to \( \lambda_i \).

Definition 3.2. We define the composition map to be the induced composite:

\[
\Omega_{\varsigma_1}(M_0, M_1) \wedge \cdots \wedge \Omega_{\varsigma_k}(M_{k-1}, M_k) \to \Sigma(M_{0, \ldots, k})^{-\lambda_i} \to \Sigma(M_0 \times M_k)^{-\varsigma} = \Omega_{\varsigma}(M_0, M_k).
\]

We leave it to the reader to check that composition as defined above is homotopy associative. In fact, this composition has more structure. The question of how structured this associative composition is will be addressed in a later document.
Remark 3.3. We may interpret the above map geometrically as follows: Let \( Y = L_1 \times L_2 \times \cdots \times L_k \) be a product of stable lagrangians immersed in \( \overline{M}_0 \times (\overline{M}_1 \times \cdots \times \overline{M}_k) \). Assume that \( X \subset Y \) is the transverse intersection of \( Y \) along \( \overline{M}_0 \times \Delta(M_k)^{k-1} \times M_k \). Hence we get a factorization with rows being given by the Pontrjagin–Thom construction:

\[
\begin{array}{ccc}
Y^{-\tau(Y)} & \longrightarrow & X^{-\tau(X)} \\
\downarrow & & \downarrow \\
\Omega_{\zeta}(M_0, M_1) \wedge \cdots \wedge \Omega_{\zeta}(M_{k-1}, M_k) & \longrightarrow & \mathbb{S}(M_0, \ldots, k)^{-\lambda_i}
\end{array}
\]

It is easy to see that \( X \) supports a lagrangian immersion into \( \overline{M}_0 \times M_k \), represented by the composite map:

\[ [X] : S^0 \longrightarrow X^{-\tau(X)} \longrightarrow \mathbb{S}(M_0, \ldots, k)^{-\lambda_i} \longrightarrow \mathbb{S}(\overline{M}_0 \times M_k)^{-\zeta} = \Omega_{\zeta}(M_0, M_k). \]

Now let \( \Omega_{\zeta}^\infty(M, N) \) denote the infinite loop space of the Thom spectrum. Then from the geometric standpoint on the symplectic category, it is more natural to consider the unstable composition map:

\[
\Omega_{\zeta_i}(M_0, M_1) \times \cdots \times \Omega_{\zeta_k}(M_{k-1}, M_k) \longrightarrow \Omega_{\zeta}^\infty(M_0, M_k)
\]

By applying \( \pi_0 \) to this morphism, and invoking the above observation, we see that the definition of composition is faithful to Weinstein’s definition of composition in the symplectic category.

Claim 3.4. Let \( M \) be a compact manifold, and let \([id]\) denote the map \([id] : S^0 \to \Omega_{\zeta}(M, M)\) corresponding to the diagonal embedding \( \Delta : M \to \overline{M} \times M \). Then \([id]\) is indeed the identity for the composition defined above.

Proof. Given two manifolds \( M, N \), let \( \Delta \subset \overline{M} \times M \) is the diagonal representative of \([id]\) as above. Observe that \( \overline{N} \times \Delta(M) \times M \) is transverse to \( \overline{N} \times M \times \Delta \) inside \( \overline{N} \times M \times \overline{M} \times M \). They intersect along \( \overline{N} \times \Delta^3(M) \), where \( \Delta^3(M) \subset M \times \overline{M} \times M \) is the triple (thin) diagonal. Hence we get a commutative diagram:

\[
\begin{array}{ccc}
\Omega_{\zeta}(N, M) \wedge S^0 & \longrightarrow & \Omega_{\zeta}(N, M) \wedge \Delta^{-\tau(M)} \\
\downarrow & & \downarrow \\
\Omega_{\zeta}(N, M) \wedge \Omega_{\zeta}(M, M) & = & \Omega_{\zeta}(N, M)
\end{array}
\]

where the right vertical map is composition, and the left vertical map is the Pontrjagin–Thom collapse over the inclusion map \( \overline{N} \times M = \overline{N} \times \Delta^3(M) \longrightarrow \overline{N} \times M \times \Delta \). Now consider the following factorization of the identity map:

\[
\overline{N} \times M = \overline{N} \times \Delta^3(M) \longrightarrow \overline{N} \times M \times \Delta \longrightarrow \overline{N} \times M
\]

where the last map is the projection onto the first two factors. Performing the Pontrjagin–Thom collapse map over this composite shows that the following composite is the identity map:

\[
\Omega_{\zeta}(N, M) \wedge S^0 \longrightarrow \Omega_{\zeta}(N, M) \wedge \Delta^{-\tau(M)} \longrightarrow \Omega_{\zeta}(N, M).
\]

This shows that right multiplication by \([id] : S^0 \to \Omega_{\zeta}(M, M)\) induces the identity map on \( \Omega_{\zeta}(N, M) \). A similar argument works for left multiplication. \( \square \)
Remark 3.5. The description of the identity morphism \([\text{id}] : S^0 \to \Omega_\zeta(M, M)\) requires that the object \(M\) be compact. The price of extending \(S\) to include arbitrary symplectic manifolds as objects, is that we simply lose the identity morphisms for non-compact objects. Notice that the construction of \(\Omega_\zeta(M, N)\) does not require \(M\) or \(N\) to be compact. Indeed, for arbitrary symplectic manifolds \(M, N\), the infinite loop space underlying \(\Omega_\zeta(M, N)\) is a valid moduli space of stable immersed (compact) lagrangian submanifolds in \(\overline{M} \times N\). Recall that we have a canonical equivalence: \(\Omega_\zeta(M \times \mathbb{C}, N) = \Sigma^{-1} \Omega_\zeta(M, N)\). The same holds on replacing \(N\) by \(N \times \mathbb{C}\). Therefore, up to natural equivalence, the morphism spectra factor through the equivalence on symplectic manifolds defined in the introduction. We will state explicitly when our manifolds are assumed to be compact.

4. INTERNAL STRUCTURE OF THE STABLE SYMPLECTIC CATEGORY

Let \(\Omega_\zeta(\ast, \ast)\) be denoted by \(\Omega\), and we think of it as the “coefficient” spectrum. From the previous section, we know that \(\Omega = (U/O)^{-\zeta}\) is a commutative ring spectrum. This spectrum has been studied in \([2]\) (Section 2). In particular, \(\pi_* \Omega = \mathbb{Z}/2[x_{2i+1}, i \neq 2^k - 1]\).

In addition, the ring \(\pi_* \Omega\) can be detected as a subring of \(\pi_* M\). It follows that \(\Omega\) is a generalized Eilenberg–Mac Lane spectrum over \(H(\mathbb{Z}/2)\). Since it acts on all morphism spectra \(\Omega_\zeta(M, N)\), we see that all the spectra \(\Omega_\zeta(M, N)\) are also generalized Eilenberg–Mac Lane spectra (see the Appendix). In addition, the composition map is defined in the category of \(\Omega\)-module spectra. In the special case \(M = N\), notice that \(\Omega_\zeta(M, M)\) is a ring spectrum \(^2\). In general, this ring spectrum will not have a unit unless \(M\) is compact. For an arbitrary pair \(M, N\), the spectrum \(\Omega_\zeta(M, N)\) is a (left/right) module over the respective ring spectra. If \(M\) is a compact symplectic manifold, then we may smash with the identity \([\text{id}]\) to obtain a ring map \(\Omega \to \Omega_\zeta(M, M)\). It can be shown that the action of \(\Omega\) on \(\Omega_\zeta(M, N)\) factors through this map.

Remark 4.1. One has an oriented version of this category \(sS\), obtained by replacing \(BO\) by \(BSO\). All definitions go through in this setting verbatim. However, the “coefficients” \(s\Omega\) is no longer a generalized Eilenberg–Mac Lane spectrum. Its rational homology (or stable homotopy) is easily seen to be an exterior algebra:

\[\pi_* s\Omega \otimes \mathbb{Q} = \Lambda(y_{4i+1})\]

The following results are easy consequences of the definitions:

Claim 4.2. Given arbitrary symplectic manifolds \(M\) and \(N\), there is a natural decomposition of \(\Omega_\zeta(M, N)\) induced by the composition map:

\[\Omega_\zeta(M, \ast) \wedge_\Omega \Omega_\zeta(\ast, N) = \Omega_\zeta(M, N)\]

In particular, arbitrary compositions can be canonically factored, and computed by applying the following composition map internally:

\[\Omega_\zeta(\ast, N) \wedge_\Omega \Omega_\zeta(N, \ast) \to \Omega\]

Claim 4.3. Let \(M\) be a compact symplectic manifold. Then the spectrum \(\Omega_\zeta(M, \ast)\) is canonically equivalent to \(\text{Hom}_{\Omega}(\Omega_\zeta(\ast, M), \Omega)\). In other words, \(\Omega_\zeta(\ast, M)\) is conjugate dual.

\(^2\) See theorems 4.5 and 8.4 for an interpretation of \(\Omega_\zeta(M, M)\).
Proof. Consider the composition map: $\Omega_\zeta(\ast, M) \wedge \Omega_\zeta(M, \ast) \to \Omega$. Taking the adjoint of this map yields the map we seek to show is an equivalence:

$$\Omega_\zeta(M, \ast) \to \text{Hom}_\Omega(\Omega_\zeta(\ast, M), \Omega).$$

To construct a homotopy inverse to the above map, one uses the identity morphism:

$$\text{Hom}_\Omega(\Omega_\zeta(\ast, M), \Omega) \wedge S^0 \to \text{Hom}_\Omega(\Omega_\zeta(\ast, M), \Omega) \wedge \Omega \Omega_\zeta(M, M) \to \Omega_\zeta(M, \ast),$$

where the last map is evaluation, once we identify $\Omega_\zeta(M, M)$ with $\Omega_\zeta(M, \ast) \wedge \Omega_\zeta(\ast, M)$ using the previous claim. Details are left to the reader.

□

Remark 4.4. This is really Poincaré duality in disguise that says the dual of the Thom spectrum of a bundle $\xi$ over $M$ is the Thom spectrum of $-\xi - \tau$, where $\tau$ is the tangent bundle of $M$. So the bundle that is “self dual” is the bundle $\xi$ so that $\xi = -\xi - \tau$, i.e. $2\xi = -\tau$. The bundle $-\zeta$ is the universal bundle that satisfies this condition.

Theorem 4.5. For arbitrary symplectic manifolds $M$ and $N$, there are canonical equivalences of $\Omega$-module spectra:

$$\Omega_\zeta(\ast, M \times N) = \Omega_\zeta(M \times N, \ast) = \Omega_\zeta(\overline{N}, \overline{M}) = \Omega_\zeta(M, N).$$

Furthermore if $M$ is compact, then using duality on $M$, we have an equivalence:

$$\Omega_\zeta(M, N) = \text{Hom}_\Omega(\Omega_\zeta(\ast, M), \Omega_\zeta(\ast, N))$$

which is compatible with composition in $\mathbb{S}$. In particular, for a compact manifold $M$, the ring spectrum $\Omega_\zeta(M, M)$ has the structure of an endomorphism algebra $^3$.

$$\Omega_\zeta(M, M) = \text{End}_\Omega(\Omega_\zeta(\ast, M)).$$

The above identification suggests that $\Omega_\zeta(M, M)$ should be viewed as the homotopical “algebra of observables” of $M$. All this structure holds for $s\Omega_\zeta(M, M)$ as well. Furthermore, in the unoriented case, the above theorem may be strengthened (see theorem 8.4).

5. The Metaplectic Category

Kähler manifolds admit a holomorphic version of geometric quantization. This quantization is the space of square integrable holomorphic sections of the line bundle given by $L \otimes \sqrt{\text{det}}$, where $L$ is the prequantum line bundle, and $\sqrt{\text{det}}$ is a choice of square root of the volume forms (called a metaplectic structure). A derived version of this quantization should in principle depend on all the cohomology groups of this line bundle. Now the Dolbeaut $\overline{\partial}$-complex computing the cohomology of holomorphic bundles agrees with the Spin$^c$ Dirac operator. On incorporating the metaplectic structure into the picture, the complex agrees with the Spin-Dirac operator. We therefore observe that almost complex Spin manifolds support a Dirac operator which generalizes the Dolbeaut complex twisted by the square root of the volume forms. This may suggest defining a derived version of the geometric quantization of a symplectic manifold with a Spin structure as the $L^2$ index of the Dirac operator with coefficients in the prequantum line bundle. For compact manifolds this is simply the $\hat{A}$ genus with values in the prequantum line bundle. We take this as good motivation to explore the nature of the stable symplectic category supporting symplectic manifolds with this structure.

$^3$In fact, we show in section 7 that the symplectomorphism group of $M$ maps to the units in this algebra.
In this section we will extend the constructions given in the previous sections to the metaplectic category. The objects in this category will be symplectic manifolds endowed with a compatible metaplectic structure. Informally speaking, a compatible metaplectic structure is a compatible complex structure endowed with a square root of the determinant line bundle. Let us now formalize the concept of a metaplectic structure. Let the classifying space of the metaplectic group \( \tilde{U} \), denoted by \( B\tilde{U} \), be defined via the fibration:

\[
B\tilde{U} \longrightarrow BU \longrightarrow K(\mathbb{Z}/2, 2)
\]

with the second map being the mod-2 reduction of the first Chern class. By definition, \( B\tilde{U} \) supports the square-root of the determinant map \( \sqrt{det} : B\tilde{U} \longrightarrow BS^1 \). The complexification map \( BSO \longrightarrow BU \) lifts to a unique map \( BSpin \longrightarrow B\tilde{U} \). We may describe these lifts as a diagram of fibrations:

\[
\begin{array}{cccc}
\tilde{U}/Spin & \longrightarrow & U/SO & \longrightarrow \ K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2) \\
\downarrow & & \downarrow & \\
BSpin & \longrightarrow & BSO & \longrightarrow \ K(\mathbb{Z}/2, 2) \\
\downarrow w_2 & & \downarrow 0 & \\
B\tilde{U} & \longrightarrow & BU & \longrightarrow \ K(\mathbb{Z}/2, 2) \\
\end{array}
\]

An easy calculation shows that the mod-2 cohomology \( H^*(U/SO, \mathbb{Z}/2) \) is an exterior algebra on generators \( \{\sigma, w_2, w_3, \ldots\} \), where \( \sigma \) is the class that transgresses to \( c_1 \) in the mod-2 Serre spectral sequence, and \( w_i \) are the classes that are given by the corresponding restrictions from \( H^*(BSO, \mathbb{Z}/2) \). In addition, the class \( U/SO \longrightarrow K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2) \) is represented by the product \( \sigma \times w_2 \). Hence \( \tilde{U}/Spin \) is the corresponding cover of \( U/SO \).

**Remark 5.1.** The kernel of the map \( \tilde{U} \longrightarrow U/O \) is given by a group \( \tilde{O} \) called the Metalinear group (see [12] Section 7.2). The classifying space \( B\tilde{O} \) is described by a fibration:

\[
\begin{array}{cccc}
B\tilde{O} & \longrightarrow & BO & \longrightarrow \ K(\mathbb{Z}/2, 2) \\
\downarrow & & \downarrow w_2^2 & \\
B\tilde{U} & \longrightarrow & BU & \longrightarrow \ K(\mathbb{Z}/2, 2) \\
\end{array}
\]

The group \( \tilde{O} \) can easily be seen as a split \( \mathbb{Z}/4 \) extension of \( SO \). The map \( B\tilde{O} \longrightarrow B(\mathbb{Z}/4) \) is sometimes called the Maslov line bundle. Notice also that \( BSpin \) can be seen as the cover of \( B\tilde{O} \) given by prescribing a trivialization of the Maslov line bundle, followed by a Spin structure.

We may now define the stable metaplectic category \( \tilde{S} \) along the same lines as before. The objects of \( \tilde{S} \) are symplectic manifolds with a lift of their stable tangent bundle to \( B\tilde{U} \). The morphism spectra \( \tilde{\Omega}_*(M, N) \) are defined as

\[
\tilde{\Omega}_*(M, N) := \tilde{S}(M \times N)^{-\zeta}
\]
where $\zeta$ is the $n + m$ dimensional bundle defined by the pullback:

$$
\begin{array}{ccc}
\tilde{S}(M \times N) & \xrightarrow{\zeta} & Z \times B\text{Spin} \\
\downarrow \zeta & & \downarrow \\
M \times N & \xrightarrow{\tau} & Z \times B\tilde{U}
\end{array}
$$

where $\tau$ denotes the $2(m + n)$-dimensional tangent bundle of the product metaplectic manifold $M \times N$. Composition is defined analogously. As before, all structure maps in the category are module maps over the coefficient spectra $\tilde{\Omega} = (\tilde{U}/\text{Spin})^{-\zeta}$.

6. Symbols and Thom classes

Of particular interest in the theory of geometric quantization is the metaplectic category. Recall that one may motivate a derived version of quantization of a metaplectic manifold $M$ as the $L^2$-index of the Dirac operator with values in the prequantum line bundle $\mathcal{L}$. This is motivated by the fact that for Kähler manifolds, this index is the holomorphic Euler characteristic of the prequantum line bundle twisted by the square root of the volume forms. Under suitable vanishing conditions this Euler characteristic reduces to the space of square integrable holomorphic sections, which is precisely the classical notion of the geometric quantization of a Kähler manifold. The propagators in geometric quantization are typically constructed as integral operators with kernels that are supported on lagrangian submanifolds (see [12] Chapter 6). These kernels are flat sections of the prequantum line bundle restricted to the lagrangian submanifold. We shall explore suitable scalar extensions of the metaplectic category along flat line bundles with the aim of constructing an algebraic receptacle for the propagators.

The category of symbols, and the symbol map:

In this section, we construct a family of categories $\mathbb{S}(B(h))$, indexed by a nonzero parameter $h \in \mathbb{R}^*$, which should be thought of the category of symbols for immersed lagrangians in symplectic manifolds. This family admits a canonical projection to the trivial family of symplectic categories $\mathbb{S} \times \mathbb{R}^*$. We also construct a non-trivial section to this projection, which we call the symbol map. In what follows, we make our constructions with the symplectic category $\mathbb{S}$, with the understanding that all results hold for the categories $s\mathbb{S}$ and $\bar{\mathbb{S}}$. Our constructions are arguably not as geometric as one would like; though they do seem to suggest that there is perhaps an interesting underlying geometry.

Let $B$ denote the classifying space of the group $S^1$ considered as a discrete group. So $B = K(S^1, 1)$. Notice that $B$ classifies unitary line bundles with a flat connection, and one has an extension of commutative topological groups:

$$
K(\mathbb{R}, 1) \longrightarrow B \longrightarrow BS^1.
$$

This extension is classified by a cocycle $c_1 \in H^2(BS^1, \mathbb{R})$ which is the universal first Chern class with real coefficients. This is the restatement of the fact that a line bundle admits a flat connection if its curvature is an exact form. Any two choices of a flat connection on a line bundle over a base $B$ differ by a closed form representing a class in $H^1(B, \mathbb{R})$. We may also define the extension $B(h)$ to be the one classified by the cocycle $\frac{1}{h}c_1$. 

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**Definition 6.1.** For $h \in \mathbb{R}^*$, define $S(B(h))$ to be the stable category with the same objects as $S$, and morphisms given by $\Omega_c(M, N) \wedge B(h)_+$. Since $B(h)$ is a commutative topological group, its suspension spectrum is a commutative ring spectrum. This allows us to define composition exactly as before. All structure maps in the category $S(B(h))$ are defined in the category of $\Omega(B(h))$-module spectra, where the coefficient spectrum $\Omega(B(h))$ is the spectrum $(U/O)^{-\xi} \wedge B(h)_+$.

Let $S \times \mathbb{R}^*$ denote the trivial family of categories. Notice that there is a canonical projection functor: $\rho : S(B(h)) \longrightarrow S \times \mathbb{R}^*$. The next step is to define a suitable section of $\rho$ which one may call the symbol $\sigma$ for the symplectic category.

Fix a symplectic manifold $M$ with a compatible complex structure. Assume $h$ is a nonzero constant with the property that $h[\omega]$ is an integral cohomology class. Fix a lift of $h[\omega]$ to $H^2(M, \mathbb{Z})$, and let $L(h)$ denote the “prequantum” line bundle classified by that lift. Now recall our assumption that the cohomology class of the symplectic form $\omega$ is a scalar multiple of the first Chern class of $M$. Hence the class $h[\omega] \in H^2(M, \mathbb{R})$ restricts trivially along the map $\xi$ (see below). Hence a lift $\hat{L}(h)$ exists:

$$
\begin{array}{ccc}
S(M) & \overset{\hat{L}(h)}{\longrightarrow} & B(h) \\
\downarrow \xi & & \downarrow \\
M & \overset{L(h)}{\longrightarrow} & BS^1.
\end{array}
$$

Let us also make a choice of lift $\hat{L}(h) = \hat{L}(h)(M)$ for the manifold $M$, and fix that choice henceforth. Assume that the choice of the lift $\hat{L}(h)(M)$ for the conjugate manifold is given by composing $\hat{L}(h)(M)$ with the formal negative map on $B(h)$.

Consider the special case of the symplectic manifold $C$. Let us choose the trivial lift to $B(h)$ of the prequantum line bundle. One map extend it to the trivial lift $\hat{L}(h)$ of the prequantum line bundle $\Lambda^k(C^k)$ over $C^k$. We denote by $L_k(h)$ the prequantum line bundle $L(h) \otimes \Lambda^k(C^k)$ on $M \times C^k$. By the above constructions, we have a natural sequence of lifts $\hat{L}_k(h)$ as shown below:

$$
\begin{array}{ccc}
S(M \times C^k) & \overset{\hat{L}_k(h)}{\longrightarrow} & B(h) \\
\downarrow \xi & & \downarrow \\
M \times C^k & \overset{L_k(h)}{\longrightarrow} & BS^1.
\end{array}
$$

Composing $\hat{L}_k(h)$ with the diagonal map yields a sequence of natural maps of spectra:

$$
\sigma_k(h)(M) : \Sigma^{m+k}S(M \times C^k)^{-\xi_k} \longrightarrow \Sigma^{m+k}S(M \times C^k)^{-\xi_k} \wedge B(h)_+.
$$

Define $\sigma(h)(M) : S(M)^{-\xi} \longrightarrow S(M)^{-\xi} \wedge B(h)_+$ to be the colimit of this sequence.

Given two symplectic manifolds $M$ and $N$ with the property that $h[\omega]$ is integral for both, the above construction gives us the symbol map on the spectra $\Omega_c(M, \ast)$ and $\Omega_c(\ast, N)$. Now using the canonical factorization: $\Omega_c(M, \ast) \wedge_\Omega \Omega_c(\ast, N) = \Omega_c(M, N)$, we get a symbol:

$$
\sigma(h)(M, N) : \Omega_c(M, N) \longrightarrow \Omega_c(M, N) \wedge B(h)_+.
$$

We leave it to the reader to check that this map respects composition.
Now define a category $\mathcal{S}(h^{-1}\mathbf{Z})$ whose objects are pairs $(M, h)$, where $M = (M, \omega)$ is a symplectic manifold with the property that $h[\omega]$ is endowed with an integral lift. Morphisms are defined as $\Omega_\zeta(M, N)$ if $(M, h)$ and $(N, h)$ are objects with the same scalar $h$.

**Definition 6.2.** Define the symbol functor $\sigma : \mathcal{S}(h^{-1}\mathbf{Z}) \longrightarrow \mathcal{S}(\mathcal{B}(h))$ which sends each object to itself, and on morphisms, is given by:

$$
\sigma(h)(M, N) : \Omega_\zeta(M, N) \longrightarrow \Omega_\zeta(M, N) \wedge \mathcal{B}(h)_+.
$$

**Remark 6.3.** We point out that there were choices made in the construction of the symbol. One may parametrize the choices as a torsor over the 1-cocycles on $\mathbb{S}$ with values in the the cohomology groups $H^1(\overline{M} \times N, \mathbb{R})$. Hence, one should interpret symbols as a notion of a flat connection on the symplectic category.

**Example 6.4.** Returning to the example of $M = T^*X$, for a compact manifold $X$. Any symbol map is given by:

$$
\sigma : \Omega \wedge X^{-\tau(X)} \longrightarrow \Omega(\mathcal{B}) \wedge X^{-\tau(X)} = \text{Map}(X, \Omega(\mathcal{B})).
$$

This map destabilizes to the map that sends a stable lagrangian $L$ to the parametrized family of virtual dimension zero stable lagrangians along $\pi : L \to X$, with values in a flat bundle. Of course, we are describing the push forward of the (flat) restriction of the prequantum line bundle on $L$, along the map $\pi$.

**Thom classes:**

The next item on the agenda is to construct Thom classes. Notice that the Thom spectra $\Omega_\zeta(M, N)$ admit canonical maps to $\Sigma^{-(m+n)} \text{MO}$ that classify the map induced by the virtual bundle $-\zeta$. Similarly, the oriented and metaplectic categories $\mathbb{S}, \tilde{\mathbb{S}}$ admit canonical maps to MSO and MSpin respectively. Continuing to work with $\mathbb{S}$ for simplicity, let $E$ now be a spectrum with the structure of a commutative algebra over the spectrum $\text{MO}$. This allows us to obtain $\Omega$-equivariant Thom classes:

$$
E(M, N) : \Omega_\zeta(M, N) \longrightarrow \Sigma^{-(m+n)} \text{MO} \longrightarrow \Sigma^{-(m+n)} E.
$$

**Remark 6.5.** Given a choice of Thom classes, notice that any object $N$ in $\mathbb{S}$ has an induced $E$ orientation via the composite map $N^{−\tau} \to \Omega_\zeta(N, N) \to \Sigma^{-2n} E$, induced by a lift of the diagonal inclusion $N \to \overline{N} \times N$. The multiplicative property of the Thom classes can be written as follows. By restricting $-(\zeta(M, N) \oplus \zeta(N, P))$ over $\overline{M} \times \Delta(N) \times P$, we have the equality of Thom classes given by: $E(M, N) \wedge E(N, P) = [N] \wedge E(M, P)$, where $[N] \in E^{-2n}(N^{−\tau})$ is the $N$-orientation.

The Thom isomorphism theorem is now a formal consequence of the definitions. Consider the diagonal map: $\Omega_\zeta(M, N) \longrightarrow \Omega_\zeta(M, N) \wedge (\overline{M} \times N)_+$, where $(\overline{M} \times N)_+$ denotes the space $\overline{M} \times N$ with a disjoint base point. This allows us to construct Thom maps in $E$-homology and cohomology, given by capping with the Thom class, and cupping with it respectively. The Thom isomorphism now follows by an easy argument:

**Claim 6.6.** Let $M$ and $N$ be any two symplectic manifolds of dimension $2m$ and $2n$ resp. Given $\Omega$-equivariant Thom classes as above, there are canonical Thom isomorphisms:

$$
\pi_*(\Omega_\zeta(M, N) \wedge_\Omega E) = E^{s+m+n}(\overline{M} \times N), \quad \pi_* \text{Hom}_\Omega(\Omega_\zeta(M, N), E) = E^{-s+m+n}(\overline{M} \times N).
$$

The following is now an easy consequence of 4.5:
Theorem 6.7. There exists a representation of the symplectic category in the category of $\pi_* E$-modules:
\[ q : E_{*+m}(M) \otimes \pi_* \Omega_\zeta(M, N) \to E_{*+n}(N). \]
In addition, one has an intersection pairing (which is non-degenerate for compact $M$):
\[ E_{*+m}(M) \otimes E_{*+m}(M) \to E_* . \]
This pairing, along with the above representation, induces a structure of a topological field theory on the category $S$. All of this structure also holds for the oriented and metaplectic case.

Remark 6.8. Given an $MO$-algebra $E$ as above, we may extend the category of symbols along
\[ \Omega(B(h)) = \Omega \wedge B(h)_+ \to E \wedge B(h)_+ := E(B(h)). \]
It follows from the Thom isomorphism that the symbol of a lagrangian $L$ in $\overline{M} \times N$ belongs to $E(B(h))_{m+n}(\overline{M} \times N)$. Furthermore, if $M$ and $N$ are assumed to be compact symplectic manifolds, then the symbol may be seen as an element in $E(B(h))_{m+n}(\overline{M} \times N)$ given by the composite:
\[ \overline{M} \times N \to L^{\tau(L)} \to \Sigma^{m+n} E(B(h)), \]
where the first map is the Pontrjagin–Thom collapse, and the second map is given by the Thom class for $\tau(L)$ in $E$-cohomology, smashed with the prequantum line bundle on $L$.

Derived Geometric Quantization as a 2-trace of the symbol:

Motivated by the introductory discussion, let us now specialize to the case of the metaplectic category $\tilde{S}$. In this section, we shall extend coefficients along the $MSpin$ module $KU$. Let us begin by recalling the symbol map from the previous section:
\[ \sigma : S(h^{-1}Z) \to S(B(h)). \]
Notice that there is also the canonical “identity” functor that factors through the map $S^0 \to B(h)_+$:
\[ \text{Id} : S(h^{-1}Z) \to S(B(h)). \]

Definition 6.9. A categorical 2-trace of $\sigma$ is a natural transformation $H : \text{Id} \Rightarrow \sigma$. In other words $H$ is given by a collection of maps $H(M) : S^0 \to \Omega_\zeta(M, M) \wedge B(h)_+$, so that the following diagram commutes:
\[ \begin{array}{ccc}
\Omega_\zeta(M, N) & \xrightarrow{\text{Id} \wedge H(N)} & \Omega_\zeta(M, N) \wedge \Omega \Omega_\zeta(N, N) \wedge B(h)_+ \\
\downarrow^{H(M) \wedge \sigma} & & \downarrow^{\ast} \\
\Omega_\zeta(M, M) \wedge \Omega_\zeta(M, N) \wedge B(h)_+ & \xrightarrow{\ast} & \Omega_\zeta(M, N) \wedge B(h)_+ ,
\end{array} \]
where the bottom horizontal, and right vertical maps are given by composition denoted by $\ast$. It is easy to see from the definition that on the subcategory generated by compact objects, a canonical example of the 2-trace of the symbol is given by the map $H(M) = [id] \cap (1 \otimes L) : S^0 \to \Omega_\zeta(M, M) \wedge BS^1$. Notice that the line bundle $1 \otimes L$ is not flat when restricted to the diagonal lagrangian $[id] : \Delta \subset \overline{M} \times M$, and hence we are forced to take values in the (topological) line bundle $BS^1$. We therefore have:
Theorem 6.10. Let $S(BS^1)$ denote the category $S \wedge BS^1_+$ supporting a canonical map from $S(B(h))$. Then on extension to $S(BS^1)$, the class $\mathcal{H}(M) = [id] \cap (1 \otimes L)$ is a canonical 2-trace of the symbol on the subcategory generated by compact objects. In particular, working with the meta-plectic category, and extending coefficients along the ring map $\tilde{\Omega} \wedge BS_1^+ \to KU$, the canonical 2-trace $\mathcal{H}(M) \in KU_{2m}(\overline{M} \times M)$ may be identified with the derived quantization of $M$ given by the Dirac operator on $M$ twisted by the prequantum line bundle $L$.

Proof. The only thing that requires proof is to show that on extending coefficients to $KU$, the 2-trace $\mathcal{H}(M)$ can be canonically identified with the derived quantization of $M$. For this, recall that the extension of $\mathcal{H}(M)$ is given by the composite:

$$S^0 \xrightarrow{\tau(M)} M^{-\tau(M)} \xrightarrow{\tilde{\Omega}_\zeta(M, M) \wedge \tilde{\Omega}} KU = \Sigma^{-2m}(\overline{M} \times M)_+ \wedge KU,$$

where the first two maps represent $M$ as the diagonal lagrangian in $\overline{M} \times M$ twisted by the class $1 \otimes L$, and the last map is the Thom isomorphism. Projecting further to $\Sigma^{-2m} KU$, we get the $\hat{A}$-genus of $M$ twisted with $L$, which was our definition of derived quantization.

$\square$

Remark 6.11. Consider the special case of $M = N$, and the map given by $\mathcal{H}(M) * \sigma$:

$$\tilde{\Omega}_\zeta(M, M) \xrightarrow{\Sigma^{-2m} KU \wedge(\overline{M} \times M)_+}.$$ 

One may restrict attention to the multiplicative monoid given by components of the moduli space: $\Omega^\infty(\tilde{\Omega}_\zeta(M, M))$ consisting of those stable lagrangian immersions $L \to \overline{M} \times M$ that map the (K-theory) fundamental class of $L$ to that of the diagonal $\Delta$. Then the image of the map $\mathcal{H}(M) * \sigma$ restricted to this subspace is the class $\mathcal{H}(M) \in \Omega^{\infty+2m}(KU \wedge(\overline{M} \times M)_+)$. In particular, the lagrangians in these components “act” on the derived quantization $\mathcal{H}(M)$. 

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7. A second Stabilization

This section is very different in flavor from the previous sections. It is perhaps aimed at a more homotopical audience and the geometric reader may wish to skip it.

Working for simplicity with the category $\mathbb{S}$, let $(M, \omega)$ be a compact object. Recall from 4.5 that $\Omega(\zeta(M, M)) = \text{End}_\Omega(\Omega(\zeta(M, M)))$.

Now let $GL \Omega(\zeta(M, M)) = \text{Aut}_\Omega(\Omega(\zeta(M, M)))$ be defined as the components that induce invertible $\pi_0(\Omega)$-module maps in homotopy. Since $\Omega(\zeta(\ast, M))$ is a finite cellular $\Omega$-module, we may define a “second stabilization map”:

$$Q(M) : BGL \Omega(\zeta(M, M)) \longrightarrow \Omega^\infty K(\Omega)$$

where $K(\Omega)$ denotes Waldhausen’s Algebraic K-theory spectrum of the commutative ring spectrum $\Omega$. It is clear from the definition that the second stabilization can be defined for the categories $s\mathbb{S}$ and $\tilde{\mathbb{S}}$ as well.

Remark 7.1. Notice also that the map $Q(M)$ may be extended to $\text{THH}(\Omega)$:

$$Q(M) : BGL \Omega(\zeta(M, M)) \longrightarrow \Omega^\infty \text{THH}(\Omega) = \Omega^\infty LB(U/O)^{\circ \zeta} = \Omega^\infty ((Sp/U)_+ \wedge \Omega),$$

where we have used [4] to identify $\text{THH}(\Omega)$. In the unoriented case this seems like a tractable map to try and understand since we have a good handle on the spectra $\Omega(\zeta(M, M))$. In addition, we also know the homotopy of $\text{THH}(\Omega)$:

$$\pi_* \text{THH}(\Omega) = \mathbb{Z}/2[x_{2i+1}, y_{4j+2}, i \neq 2^k - 1].$$

Let us now concentrate on the oriented category $s\mathbb{S}$. The spectrum $s\Omega$ is a connective spectrum with $\pi_0 s\Omega = \mathbb{Z}$. Let us consider the fibration:

$$K(\pi) \longrightarrow K(s\Omega) \longrightarrow K(\mathbb{Z})$$

where $\pi$ is the fiber of the zero-th Postnikov section: $s\Omega \longrightarrow H\mathbb{Z}$.

Claim 7.2. Let $\overline{K}(s\Omega)$ denote the cofiber of the canonical map $K(S^0) \rightarrow K(s\Omega)$. Then rationally, the spectrum $K(\pi)$ is equivalent to $\overline{K}(s\Omega)$. In particular the above fibration admits a canonical rational splitting, and there exist polynomial classes $y_{4i+2}$ in degree $4i + 2$ such that $\pi_* K(\pi)$ is isomorphic to the augmentation ideal:

$$\pi_* K(\pi) \otimes \mathbb{Q} = \mathbb{Q}[y_{4i+2}]_{>0}.$$ 

Furthermore, rationally $\pi_* K(\pi)$ can be identified with the injective image of the canonical map:

$$\Omega^\infty \Sigma^\infty (\text{BGL}_1(s\Omega)) \longrightarrow \text{BGL}_\infty(s\Omega)^+ = \Omega^\infty K(s\Omega)$$

Proof. Since $K(S^0)$ is rationally equivalent to $K(\mathbb{Z})$, the first part of the claim in clear. Now, via the Thom isomorphism, we may identify $s\Omega$ rationally with the ring spectrum $\Sigma^\infty (U/\mathbb{SO})_+$. In particular $\pi_* s\Omega \otimes \mathbb{Q} = \Lambda(y_{4i+1})$. Now invoking results from [1], we see that $\pi_* K(\pi)$ can be identified with positive degree elements in $\text{THH}_*(U/\mathbb{SO})_+$ that are in the kernel of the Connes boundary operator. These elements are given by the augmentation ideal in $H_*(B(U/\mathbb{SO})) = \mathbb{Q}[y_{4i+2}]$. The proof of the claim is complete once we observe that the rational equivalence between $s\Omega$ and $\Sigma^\infty (U/\mathbb{SO})_+$ induces a rational equivalence between $\text{BGL}_1(s\Omega)$ and $B(U/\mathbb{SO})$. \qed
Now given a compact symplectic manifold \((M, \omega)\), there exists a homomorphism from the symplectomorphism group to the units \(s\Omega_{\zeta}(M, M)\) given by taking a symplectomorphism to its graph:

\[
\text{BGr} : \text{BSymp}(M, \omega) \longrightarrow \text{BGL } s\Omega_{\zeta}(M, M)
\]

To explicitly construct this map observe that we may construct the spectrum \(s\Omega_{\zeta}(\ast, M)\) fiberwise over \(\text{BSymp}(M, \omega)\). More precisely, consider the space \(J(M)\) consisting of pairs \((J, m)\) with \(m \in M\) and \(J\) a compatible complex structure on \((M, \omega)\). This space fibers over the space of compatible complex structures on \((M, \omega)\), with fiber \(M\). In particular, it is homotopy equivalent to \(M\). There is a natural action of the symplectomorphism group on \(J(M)\), and the total space of the associated bundle \(E_{\text{Symp}}(M, \omega) \times_{\text{Symp}} J(M)\) supports a canonical complex vector bundle extending the fiberwise bundle \(TM\). Since we may replace \(J(M)\) by \(M\) without changing the homotopy type, the space given by \(\hat{M} = E_{\text{Symp}}(M, \omega) \times_{\text{Symp}} M\) supports a complex vector bundle extending the tangent bundle of \(M\).

Consider the pullback \(s\mathcal{S}(\hat{M})\) fibering over \(\text{BSymp}(M, \omega)\):

\[
\begin{array}{ccc}
\text{BSymp}(M, \omega) & \xrightarrow{s\mathcal{S}(\hat{M})} & \mathbb{Z} \times \text{BO} \\
\downarrow & & \downarrow \\
\text{BSymp}(M, \omega) & \xrightarrow{\hat{M}} & \mathbb{Z} \times \text{BU}
\end{array}
\]

In particular, the fiberwise Thom spectrum \(s\mathcal{S}(\hat{M})^{-\xi}\) is a bundle of \(s\Omega\)-module spectra over \(\text{BSymp}(M, \omega)\), with fiber being \(s\Omega_{\zeta}(\ast, M)\). We may classify the bundle by:

\[
\text{BGr} : \text{BSymp}(M, \omega) \longrightarrow \text{BAut}_{s\Omega} s\Omega_{\zeta}(\ast, M) := \text{BGL } s\Omega_{\zeta}(M, M)
\]

We may now consider the \(K(s\Omega)\)-valued parametrized index defined as the composite:

\[
I(M) = Q(M) \circ \text{BGr} : \text{BSymp}(M, \omega) \longrightarrow \Omega^\infty K(s\Omega)
\]

By [5], the map \(I(M)\) has a factorization:

\[
\text{BSymp}(M, \omega) \longrightarrow \Omega^\infty \Sigma^\infty(\hat{M}_+) \longrightarrow \Omega^\infty A(\hat{M}) \longrightarrow \Omega^\infty K(s\Omega)
\]

where \(A(\hat{M})\) is the Waldhausen K-theory of \(\hat{M}\), the first map is the Becker–Gottlieb transfer, the second is the natural transformation between \(\Sigma^\infty(X_+)\) and \(A(X)\), and the final map is the one that classifies the stable bundle of \(s\Omega\)-spectra over \(\hat{M}\).

Using naturality, we have the following description of \(I(M)\) given by the commutative diagram with the vertical maps begin induced by the map that classifies the (rank one)
bundle of $s\Omega$-spectra over $\hat{M}$:

\[
\begin{array}{ccccccc}
\Sigma^\infty(\text{BSymp}(M, \omega)_+) & \xrightarrow{\text{tr}} & \Sigma^\infty(\hat{M}_+) & \rightarrow & A(\hat{M}) & \rightarrow & \text{K}(s\Omega) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^\infty(\text{BSymp}(M, \omega)_+) & \rightarrow & \Sigma^\infty(\text{BU}_+) & \rightarrow & A(\text{BU}) & \rightarrow & \text{K}(s\Omega) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^\infty(\text{BSymp}(M, \omega)_+) & \rightarrow & \Sigma^\infty(\text{BGL}_1(s\Omega)_+) & \rightarrow & A(\text{BGL}_1(s\Omega)) & \rightarrow & \text{K}(s\Omega)
\end{array}
\]

Let us now attempt to understand the map $\text{BU} \rightarrow \Omega^\infty\Sigma^\infty(\text{BGL}_1(s\Omega)_+)$, given by applying $\Omega^\infty\Sigma^\infty$ to the map $\text{BU} \rightarrow \text{BGL}_1(s\Omega)$ that classifies the bundle of $s\Omega$-spectra over $\text{BU}$.

**Theorem 7.3.** Let $\lambda : \text{BU} \rightarrow \text{BGL}_1(s\Omega)$ be the map above. Then its image in cohomology is the polynomial algebra generated by all the odd Newton polynomials $N_{2k+1}(c_1, c_2, \ldots, c_{2k+1})$ in the universal Chern classes.

**Remark 7.4.** We remind the reader that the Newton polynomials $N_i(\sigma_1, \sigma_2, \ldots, \sigma_i)$ are defined by (uniquely) writing the power symmetric functions $x_1^i + x_2^i + \cdots + x_i^i$ in terms of the elementary symmetric functions $\sigma_1, \sigma_2, \ldots, \sigma_i$.

**Proof.** Recall that rationally $H_*(\text{GL}_1(s\Omega))$ is identified with $H_*(\text{U}/\text{SO}_+)$, and furthermore, it is an exterior algebra on generator in degree $4k+1$, and the map induced on the level of groups by $\Omega(\lambda) : \text{U} \rightarrow \text{GL}_1(s\Omega)$ is rationally surjective. It follows that the map $\lambda : \text{BU} \rightarrow \text{BGL}_1(s\Omega)$ picks out the collection of polynomial cohomology generators in degree $4k+2$. Now since $\lambda$ can be delooped, the cohomological generators can be chosen to be primitive. It is a standard fact that the only primitive in degree $2i$ is the Newton polynomial $N_i(c_1, c_2, \ldots, c_i)$. □

The following result follows from the above theorem and claim 7.2.

**Theorem 7.5.** The parametrized index map $I(M)$ can be identified with the map given by applying the Becker–Gottlieb transfer: $[\text{tr}] : H^*(\hat{M}) \rightarrow H^*(\text{BSymp}(M, \omega))$ to a sub-algebra inside $H^*(\hat{M})$. This sub-algebra is generated by the odd Newton polynomials in the Chern classes of the fiberwise tangent bundle of $\hat{M} \rightarrow \text{BSymp}(M, \omega)$.

**Remark 7.6.** Notice that the claim 7.2 shows that we have a rational pullback diagram:

\[
\begin{array}{ccc}
\Sigma^\infty(\text{BGL}_1(s\Omega)_+) & \rightarrow & S^0 \\
\downarrow & & \downarrow \\
\text{K}(s\Omega) & \rightarrow & \text{K}(\text{Z})
\end{array}
\]

Hence we may define secondary rational (Reidemeister) invariants in dimensions $4k$, for families of symplectic manifolds $(M, \omega)$ that admit a prescribed null homotopy for the parametrized index.
The Unoriented Case:

Let us make some explicit computations in the case of the unoriented symplectic category $\mathcal{S}$. We invoke the Adams spectral sequence to compute $\pi_\ast \Omega_\zeta(M, N)$. Since $\Omega_\zeta(M, N)$ is a generalized Eilenberg–Mac Lane spectrum, the spectral sequence will collapse and we simply need to compute the primitives under the action of the dual mod-2 Steenrod algebra on $H_* \Omega_\zeta(M, N) := H_*(\mathbb{S}(M \times N)^\zeta)$.

**Remark 8.1.** Let us set some notation. All homology groups will be understood to be over $\mathbb{Z}/2$. In addition, let us use the suggestive notation $\Sigma^{-\zeta} S_\ast$ to denote the shift $\Sigma^{-\zeta} S_\ast$ for a graded module $S_\ast$.

**Theorem 8.2.** $\pi_\ast \Omega_\zeta(M, N)$ is a free $\pi_\ast \Omega$-module on a (non-canonical) generating vector space given by $\Sigma^{-\zeta} H_\ast(M \times N)$.

**Proof.** The Thom isomorphism theorem implies that the ring $H_* \Omega$ is isomorphic to $H_*(U/O)$. Under this isomorphism we also see that $H_* \Omega_\zeta(M, N)$ is isomorphic to $\Sigma^{-\zeta} H_*(\mathbb{S}(M \times N))$. Now consider the universal fibration $U/O \to BO \to BU$. It is easy to see that the Serre spectral sequence in homology for this fibration collapses leading to the fact that $H_* (\mathbb{S}(M \times N))$ is non-canonically a free $H_*(U/O)$-module on $H_\ast(M \times N)$. From this we deduce that $H_* \Omega_\zeta(M, N)$ is free $H_* \Omega$-module on the (non-canonical) vector space given by $\Sigma^{-\zeta} H_\ast(M \times N)$. An easy argument using the degree filtration shows that the generating vector space can be chosen to have trivial action of the dual Steenrod algebra. The statement of the theorem is now complete on taking primitives under this action. □

The following consequences of the above theorem are easy (compare with 4.5):

**Theorem 8.3.** There is a natural decomposition of $\pi_\ast \Omega_\zeta(M, N)$ induced by the composition map:

$$\pi_\ast \Omega_\zeta(M, *) \otimes_{\pi_\ast \Omega} \pi_\ast \Omega_\zeta(*, N) = \pi_\ast \Omega_\zeta(M, N).$$

In particular, arbitrary compositions can be canonically factored in homotopy, and computed by applying the composition map internally:

$$\pi_\ast \Omega_\zeta(*, N) \otimes_{\pi_\ast \Omega} \pi_\ast \Omega_\zeta(N, *) \longrightarrow \pi_\ast \Omega.$$

**Theorem 8.4.** Given a compact manifold $M$, the $\pi_\ast \Omega$-algebra $\pi_\ast \Omega_\zeta(M, M)$ has the structure of an endomorphism algebra:

$$\pi_\ast \Omega_\zeta(M, M) = \text{End}_{\pi_\ast \Omega} (\pi_\ast \Omega_\zeta(*, M)).$$

The Oriented Case:

Next, let us very briefly explore the structure of $s \Omega_\zeta(*, M)$ rationally. Firstly recall that $H^*(U/\text{SO}, \mathbb{Q})$ is an exterior algebra $\Lambda(y_{i+1})$. Now by Thom isomorphism, we have an equality $H^{*+m}(s\mathbb{S}(M), \mathbb{Q}) = H^*(s \Omega_\zeta(*, M), \mathbb{Q})$, where $(M, \omega)$ is a $2m$-dimensional manifold. Now consider the cohomology Serre spectral sequence for the fibration

$$U/\text{SO} \longrightarrow s\mathbb{S}(M) \longrightarrow M.$$
It is easy to see that the class $y_1 \in H^1(U/\text{SO}, \mathbb{Q})$ transgresses to a non-trivial multiple of the symplectic class $\omega$. Hence the class $y_1 \cup \omega^m$ represents the only meaningful primary characteristic class in $H^{2m+1}(s\Omega_\zeta(\ast, M), \mathbb{Q})$. Let $\theta(M)$ be the corresponding class in $H^{m+1}(s\Omega_\zeta(\ast, M), \mathbb{Q})$ under the above Thom isomorphism.

Now let $\pi : E \to B$ be a fibrating family of oriented stable lagrangians in $M$, endowed with a classifying map $f(\pi) : B \to \Omega^\infty(s\Omega_\zeta(\ast, M)).$ Then the map $f(\pi)$ factors through the Umkehr map $B_+ \to \Omega^\infty(E^{-\tau(\pi)})$ followed by the map induced by $E^{-\tau(\pi)} \to s\Omega_\zeta(M)^{-\zeta}$. It follows that

$$f(\pi)^*\theta(M) = \pi_*(y_1 \cup \omega^m),$$

where $y_1 \cup \omega^m$ denotes the pullback of the class having the same name along $E \to s\Omega_\zeta(M)$.

Some Interesting questions:

Here is a list of natural questions, some of which are quite vague or ambitious.

**Question 8.5.** Is there a universal description of the stable symplectic or metaplectic category that allows us to check if a functor defined on symplectic manifolds extends to the stable category? Notice that if $F(M)$ denotes any (stable) representation of $\mathbb{S}$, with $F(\ast) := F$ being an $\Omega$-module, then we have the action map for $F$ on the level of spectra:

$$q : \Omega_\zeta(\ast, M) \land_\Omega F(M) \rightarrow F.$$

For compact manifolds $M$, we may dualize this map to get a “symbol”:

$$F(M) \rightarrow \Omega_\zeta(\ast, M) \land_\Omega F.$$

**Question 8.6.** Describe the “Motivic Galois group”, by which we mean the rule that assigns to a commutative $\Omega$-algebra $F$, its $F$-points given by the group of multiplicative automorphisms of the monoidal functor on $\mathbb{S}$ (with values in the category of $\pi_*$ $F$-modules), and which takes a symplectic manifold $N$ to $\pi_*(\Omega(\ast, N) \land_\Omega F)$. In the oriented case of $s\Omega$, one may wish to ask the question about multiplicative automorphisms of the rational functor with $F$-points given by $\pi_*(\Omega(\ast, N) \land_\Omega F) \otimes \mathbb{Q}$.

**Question 8.7.** Recall the homomorphism from the symplectomorphism group to the units $\text{GL} \ \Omega_\zeta(\ast, M)$ given by taking a symplectomorphism to its graph:

$$\text{BGr} : \text{BSym}(M, \omega) \rightarrow \text{BGL} \ \Omega_\zeta(M, M)$$

We showed that the above map $\text{BGr}$ classifies the (twisted canonical) fibration:

$$\mathbb{S}(\tilde{M})^{-\zeta} := \text{ESym}(M, \omega)_+ \land_{\text{Sym}} \Omega_\zeta(\ast, M) \rightarrow \text{BSym}(M, \omega)$$

It can be shown that $\text{BGr}$ is null if and only if the $\Omega$-based (Atiyah–Hirzebruch) Serre spectral sequence for the above fibration collapses. It would be very interesting to know the kind of invariants the map $\text{BGr}$ detects.

**Question 8.8.** Are the degree $4n + 1$ classes in $\pi_* \text{K}(s\Omega)$ (detected in $\pi_* \text{K}(\mathbb{Z})$) related to $\pi_{4n} \text{BSym}(\mathbb{C}^k)$, for large values of $k$ and where $\text{Sym}$ denotes compactly supported symplectomorphisms?

**Question 8.9.** Is there a natural geometric structure that makes the symplectic category with coefficients in $\text{KU}(\mathcal{B}(h))$, into a “$\text{KU}$-linear” stable category? In particular, is there a geometric interpretation of the results in section 6?
REFERENCES


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