1. Introduction

Conformal Field Theory (CFT) lies at the foundation of many developments in mathematical physics in the past decades. The aim of this course is to give a basic and very mathematical introduction to elementary CFT.

One of the historical origins of this field is Polyakov’s idea in the late 60’s that CFT’s could be used to describe critical phenomena in the thermodynamic limits of various statistical models. This was brought to fruition in the seminal paper [3] where a certain minimal model was identified with the Ising model. We will describe this in detail.

A second major development was the diagonalization of the fusion rules due to Verlinde [11]. This gives rise to the famous Verlinde formula for counting conformal blocks. Applied to the Wess-Zumino-Novikov-Witten (WZNW) model this gives a dimension formula for the space of nonabelian theta functions.

The basic ingredients of a CFT are a representation of the Virasoro algebra, a state-field correspondence identifying vectors with a collection of mutually local fields, an energy-momentum tensor, and correlation functions. Locality gives rise to operator product expansions (OPE’s), and calculating them is one of the key techniques. Associativity (or crossing symmetry) of the operator algebra imposes important constraints.

The aim of any field theory is to identify the spectrum and compute all correlation functions. The remarkable thing about conformal field theory is that this can sometimes be done just by exploiting the conformal symmetry. In many of the examples we will consider, no reference to a lagrangian formulation will be made.

As a guide to references for this course, I suggest the books [5], written from a more physical perspective, combined with Kac [7], which is directed at mathematicians. I have relied on these two extensively. Another good reference is [8]. These should accompany a reading of [3].

A standard reference for physicists is Ginsparg [6]. See also the more recent lecture notes of Schellekens [10]. For relations with string theory, see [4]. The topological aspect of conformal field

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theories is also a large subject which we will probably only partially touch upon. Segal’s paper [9] is considered the standard starting point.

Of course, nothing in these notes is original. I have borrowed heavily from the references listed above.

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2. Unitary Representations of the Virasoro Algebra

2.1. Holomorphic vector fields and central extensions. We begin with the Lie algebra of holomorphic vector fields on the disk. Let

\[ l_n = -z^{n+1} \frac{d}{dz} \]

Then one computes

\[ [l_m, l_n] = (m - n) l_{m+n} \]
We call this algebra $V$. Notice that $\{l_0, l_{\pm 1}\}$ form a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. We are interested in the central extensions of $V$.

**Definition 2.1.** The Virasoro algebra $V_c$ is defined by generators $\{L_n, c\}$ satisfying

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}$$

The factor of $1/12$ is a convention. The $\mathfrak{sl}(2, \mathbb{C})$ subalgebra $\{L_0, L_{\pm 1}\}$ persists in the central extension. We shall be interested in representations of $V_c$ where the central element will be sent to a multiple of the identity. Hence, it is convenient to regard $c$ as a number with the central element $I$ understood. This number is called the *central charge* of the representation.

We have the following important result.

**Theorem 2.2.** Every non-trivial central extension of $V$ with 1-dimensional center is isomorphic to $V_c$.

2.2. **Examples.** We now consider several examples. These are key examples, the first three of which we will revisit over and over again throughout the course. They are: free bosons, free fermions, and the bc-ghost system. The latter comes up in gauge fixing. The notion of “free” in the context of field theory will be defined later. The last example, the Sugawara construction, is important in the construction of minimal models. This is perhaps a topic for the second semester.

2.2.1. **Free bosons.** The first example constructs a Virasoro representation with $c = 1$. Consider the Heisenberg algebra $\mathcal{H} = \{a_n, 1\}_{n \in \mathbb{Z}}$ satisfying

$$[a_m, a_n] = m \delta_{m+n,0}$$

Here, 1 is a central element and is naturally assumed to be the element on the right hand side above. A representation $V$ of $\mathcal{H}$ is called a *highest weight representation* if there is a vector $|0\rangle \in V$ such that

$$a_0|0\rangle = \lambda|0\rangle$$

$$a_n|0\rangle = 0 \, , \, n > 0$$

$$V = \text{span}\{a_1^{m_1} \cdots a_k^{m_k}|0\rangle : m_i \in \mathbb{Z}_+\} .$$

**Remark 2.3.** By declaring $|0\rangle$ to be a unit vector (usually written $\langle 0|0 \rangle = 1$) and $a_k = (a_{-k})^*$, we turn this representation into a unitary one.

For example,

$$\langle 0|a_k^{m_k} \cdots a_1^{m_1} a_{-1}^{m_1} \cdots a_{-k}^{m_k}|0\rangle = \|a_1^{m_1} \cdots a_{-k}^{m_k}|0\rangle\|^2 = \prod_{i=1}^{k} m_i!$$

**Proposition 2.4.** Highest weight representations of $\mathcal{H}$ are irreducible.
Proof. Define the Fock space $W = \mathbb{C}[x_1, x_2, \ldots]$ and represent $\mathcal{H}$ on $W$ by

$$a_0 = \lambda I$$
$$a_n = \frac{\partial}{\partial x_n}$$
$$a_{-n} = n x_n$$

for $n > 0$. Then it is clear that $W$ is irreducible. Given $V$ as in the statement, define the $\mathcal{H}$-equivariant homomorphism $\phi : W \to V$ by

$$\phi(p(x_1, \ldots, x_k)) = p(a_{-1}, 2a_{-2}, \ldots, na_{-n})|0\rangle$$

Since $W$ is irreducible, $\ker \phi = \{0\}$, and by assumption $\phi$ is onto. □

Then we define

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_{-k}a_{n+k} :$$

where $: :$ indicates the normal ordering

$$: a_i a_j : = \begin{cases} a_i a_j & i \leq j \\ a_j a_i & i > j \end{cases}$$

Hence,

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} a_{-k}a_{n+k}, n \neq 0$$
$$L_0 = \frac{1}{2} a_0^2 + \sum_{k>0} a_{-k}a_k$$

Proposition 2.5. With $L_n$ defined as in (2.5),

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}$$

The following is a straightforward computation.

Lemma 2.6. $[a_k, L_n] = ka_{n+k,0}$ for all $k, n$.

Proof. In doing these computations it is important to keep in mind the infinite sum vs. normal ordering. A convenient way to avoid mistakes is to introduce the function $\chi_{\varepsilon}(x) = \chi(\varepsilon x)$, where $\chi$ is the characteristic function of $(-1, 1)$, and let

$$L_n(\varepsilon) = \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_{-k}a_{n+k} : \chi_{\varepsilon}(k)$$
be the finite sum. Note that for any vector \( v \) in the Fock space \( W \), \( L_n v = \lim_{\varepsilon \to 0} L_n(\varepsilon) v \). Now for \( m \neq 0 \), we compute

\[
[L_m(\varepsilon), L_n] = \frac{1}{2} \sum_{j \in \mathbb{Z}} \{ (-j) : a_{n-j}a_{j+m} : + (j + m) : a_{-j}a_{j+m+n} : \} \chi_{\varepsilon}(j)
\]

\[
= \frac{1}{2} \sum_{j \in \mathbb{Z}} \{ (-j)a_{n-j}a_{j+m} + (j + m)a_{-j}a_{j+m+n} \} \chi_{\varepsilon}(j)
\]

We now rewrite this in terms of normally ordered sums.

\[
[L_m(\varepsilon), L_n] = \frac{1}{2} \sum_{j \in \mathbb{Z}} \{ (-j) : a_{n-j}a_{j+m} : + (j + m) : a_{-j}a_{j+m+n} : \} \chi_{\varepsilon}(j)
\]

\[
+ \frac{1}{2} \sum_{n-m > 2j} (-j)[a_{n-j}, a_{j+m}]\chi_{\varepsilon}(j)
\]

\[
+ \frac{1}{2} \sum_{-(m+n) > 2j} (j + m)[a_{-j}, a_{j+m+n}]\chi_{\varepsilon}(j)
\]

\[
= \frac{1}{2} \sum_{j \in \mathbb{Z}} \{ (-n+j)\chi_{\varepsilon}(n+j) + (m+j)\chi_{\varepsilon}(j) \} : a_{-j}a_{j+m+n} : 
\]

\[
+ \frac{1}{2} \sum_{n-m > 2j} j(j-n)\chi_{\varepsilon}(j)\delta_{m+n,0}
\]

\[
- \frac{1}{2} \sum_{-(m+n) > 2j} j(j+m)\chi_{\varepsilon}(j)\delta_{m+n,0}
\]

(suppose that \( m > 0 \))

\[
= \frac{1}{2} \sum_{j \in \mathbb{Z}} \{ (-n+j)\chi_{\varepsilon}(n+j) + (m+j)\chi_{\varepsilon}(j) \} : a_{-j}a_{j+m+n} : 
\]

\[
- \frac{1}{2} \sum_{-m \leq j < 0} j(j+m)\chi_{\varepsilon}(j)\delta_{m+n,0}
\]

\[
[L_m, L_n] = \lim_{\varepsilon \to 0}[L_m(\varepsilon), L_n] = (m-n)L_{m+n} - \frac{1}{2} \sum_{-m \leq j < 0} j(j+m)\delta_{m+n,0}
\]

and the result follows from using \( \sum_{j=1}^{m} j = (m/2)(m+1) \) and \( \sum_{j=1}^{m} j^2 = (m/6)(2m^2 + 3m + 1) \). □

We do not include a proof of the following

**Proposition 2.7.** The unitary representation of \( \mathcal{V}_1 \) on \( W \) is completely reducible.

2.2.2. Free fermions. The fermionic algebra replaces commutator relations with anticommutators

\[\{\psi_m, \psi_n\} = \psi_m\psi_n + \psi_n\psi_m = \delta_{m+n,0}\]

(2.7)

Here, \( m, n \) are either integers (“Ramond sector”) or half-integers (“Neveu-Schwarz sector”). Denote these \( \mathcal{F}_R \) and \( \mathcal{F}_{NS} \). As with the Heisenberg algebra, define vector spaces \( W_R = \mathbb{C}[\xi_1, \xi_2, \ldots] \) and
$W_{NS} = \mathbb{C}[\xi_1/2, \xi_3/2, \ldots]$, where $\xi_i$ are Grassmann variables $\{\xi_i, \xi_j\} = 0$ (alternatively, use ordinary variables and let $W$ be the exterior algebra). Represent $\mathcal{F}_{R,NS}$ on $W_{R,NS}$ by

$$
\psi_0 = \frac{1}{\sqrt{2}}(\xi_0 + \partial_{\xi_0}) \quad \text{(Ramond sector only)}
$$

$$
\psi_n = \partial_{\xi_n}
$$

$$
\psi_{-n} = \xi_n
$$

for $n > 0$.

We again define normal ordering

#### (2.8)

$$
: \psi_i \psi_j : = \begin{cases} 
\psi_i \psi_j & i \leq j \\
-\psi_j \psi_i & i > j
\end{cases}
$$

Here there is a subtlety. Define

#### (2.9)

$$
L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} k : \psi_{-k} \psi_{n+k} : + \frac{\delta_{n,0}}{16} \quad \text{(Ramond sector)}
$$

#### (2.10)

$$
L_n = \frac{1}{2} \sum_{k \in \mathbb{Z} + 1/2} k : \psi_{-k} \psi_{n+k} : \quad \text{(Neveu-Schwarz sector)}
$$

**Proposition 2.8.** With $L_n$ defined as in (2.9) or (2.10),

$$
[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{24} (m^3 - m)\delta_{m+n,0}
$$

so the central charge here is $c = 1/2$.

#### 2.2.3. $bc$-ghost system. Here we consider generators $\{b_n, c_n\}_{n \in \mathbb{Z}}$ with the following commutation relations

#### (2.11)

$$
\{b_m, c_n\} = \delta_{m+n,0}, \quad \{b_m, b_n\} = \{c_m, c_n\} = 0.
$$

Fix $\lambda$. The case of interest is when $2\lambda \in \mathbb{Z}$.

#### (2.12)

$$
L_n = \sum_{j \in \mathbb{Z}} (\lambda n - j) : b_j c_{n-j} :
$$

$$
= \sum_{j \leq -\lambda} (\lambda n - j)b_j c_{n-j} - \sum_{j \geq -\lambda + 1} (\lambda n - j)c_{n-j} b_j
$$

**Proposition 2.9.** With $L_n$ defined as in (2.12),

$$
[L_m, L_n] = (m - n)L_{m+n} + \frac{-2(6\lambda^2 - 6\lambda + 1)}{12} (m^3 - m)\delta_{m+n,0}
$$

so the central charge here is $c_\lambda = -2(6\lambda^2 - 6\lambda + 1)$. Notice that $c_\lambda = (6\lambda^2 - 6\lambda + 1)c_1$. This is related to the Mumford isomorphism on the moduli space of curves. When $\lambda = -1$, $c = -26$. This is related to the vanishing of the conformal anomaly in (bosonic) string theory.
2.2.4. *Sugawara construction.* The last example is very important. Let $g$ be a simple Lie algebra with Killing inner product $\langle \cdot, \cdot \rangle$ and orthonormal generators $T_\alpha$. Let $C^\gamma_{\alpha\beta}$ be the structure constants, and $Q$ the quadratic Casimir. Hence,

\[
[T_\alpha, T_\beta] = C^\gamma_{\alpha\beta} T_\gamma, \\
C^\gamma_{\alpha\beta} C^\delta_{\gamma\eta} = \delta_{\alpha,\beta} Q
\]

Consider now the central extension of the affine Lie algebra associated to $g$.

\[
(2.13) \quad [T^m_\alpha, T^n_\beta] = C^\gamma_{\alpha\beta} T^{m+n}_\gamma + kn\delta_{\alpha,\beta}\delta_{m+n,0}
\]

We also impose the following normal ordering

\[
T^m_\alpha T^n_\beta := \begin{cases} 
T^m_\alpha T^n_\beta & m \leq n \\
T^n_\beta T^m_\alpha & m > n \\
\frac{1}{2}(T^m_\alpha T^n_\beta + T^n_\beta T^m_\alpha) & m = n
\end{cases}
\]

Finally, let

\[
(2.14) \quad L_n = \frac{1}{Q + 2k} \sum_{\alpha=1}^{\dim g} \sum_{j \in \mathbb{Z}} : T^j_\alpha T^{n-j}_\alpha :
\]

**Proposition 2.10.** With $L_n$ defined as in (2.14),

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}
\]

where $c = 2k \dim g/(Q + 2k)$.

2.3. **Unitary representations of the Virasoro algebra.**

2.3.1. *Verma modules.* Let $V$ be a representation of $\mathcal{V}_c$. A vector $|h\rangle$ is called a highest weight vector for the representation if

\[
L_0|h\rangle = h|h\rangle, \quad L_n|h\rangle = 0 \text{ for } n > 0,
\]

where $h$ is a number. Let $M(c, h)$ be the module spanned by the descendents

\[
(2.15) \quad M(c, h) = \text{span} \{ L_{-k_1} L_{-k_2} \cdots L_{-k_t} |h\rangle : 1 \leq k_1 \leq \cdots \leq k_t \}
\]

We assume the elements in the definition of $M(c, h)$ are linearly independent, in which case $M(c, h)$ is called a Verma module. As in the examples above, $M(c, h)$ has a uniquely determined hermitian form obtained by setting $\langle h, h \rangle = 1$, and $L^*_n = L_{-n}$.

We have a decomposition $M(c, h) = \oplus_{n \in \mathbb{Z}_+} M_n(c, h)$ into subspaces of dimension $p(n) = \text{partition number of } n$, where

\[
M_n(c, h) = \text{span} \left\{ L_{-k_1} L_{-k_2} \cdots L_{-k_\ell} |h\rangle \in M(c, h) : \sum_{i=1}^{\ell} k_i = n \right\}
\]

The integer $n$ is called the level. Notice that every vector in $M_n(c, h)$ is an eigenvector of $L_0$ with eigenvalue $h+n$, and with respect to the hermitian form defined above, $M_m(c, h) \perp M_n(c, h)$ if $m \neq n$. 

It turns out that $M(c, h)$ is indecomposable but not necessarily irreducible. A vector $v \in M(c, h)$, $v \neq |h\rangle$ is called a singular vector (or null vector) if $L_n v = 0$ for all $n \geq 1$. If $v$ is an eigenvector of $L_0$, then the descendents of $v$ form a Verma submodule. Moreover,

$$\langle v, L_{-k_1} \cdots L_{-k_\ell} |h\rangle = \langle L_{k_1} v, L_{-k_2} \cdots L_{-k_\ell} |h\rangle = 0$$

Hence, $v$ is orthogonal to the whole Verma module. In particular, $\langle v, v \rangle = 0$. Let

$$N(c, h) = \{ v \in M(c, h) : \langle v, w \rangle = 0, \text{ for all } w \in M(c, h) \} .$$

Clearly, $N(c, h)$ is a submodule, Moreover,

**Lemma 2.11.** If $V \subset M(c, h)$ is a proper invariant subspace, then $V \subset N(c, h)$.

**Proof.** If $v = L_{-k_1} \cdots L_{-k_n} |h\rangle \in V$ and $w = L_{-k_1'} \cdots L_{-k_n'} |h\rangle$ with $\langle v, w \rangle \neq 0$, then $L_{k_n} \cdots L_{k_1} v \neq 0$, and so $|h\rangle \in V$. Note that every submodule must contain a $v$ of this form (i.e. a singular vector). □

Hence,

**Proposition 2.12.** The quotient space $V(c, h) = M(c, h)/N(c, h)$ is the unique irreducible highest weight representation of $\mathbb{V}_c$ with weight $h$. Moreover, it carries a non-degenerate hermitian form.

2.3.2. Kac determinant. The highest weight representation $V(c, h)$ is called degenerate if $V(c, h) \neq M(c, h)$. By Proposition 2.12, this is equivalent to the module $M(c, h)$ containing a singular vector. A singular vector exists at level $n$ if and only if the restriction of the hermitian form to $M_n(c, h)$ has a zero eigenvalue. Hence, it is equivalent to the vanishing of the determinant $\det_n(c, h)$ of the $p(n) \times p(n)$ matrix whose entries are

$$\langle h| L_{k_1'} \cdots L_{k_\ell'} |L_{-k_1} \cdots L_{-k_\ell}|h\rangle$$

$$1 \leq k_1 \leq \cdots \leq k_\ell, \quad 1 \leq k'_1 \leq \cdots \leq k'_\ell,$$

$$\sum_{i=1}^\ell k_i = \sum_{i=1}^\ell k'_i = n .$$

For example, one computes

$$\det_0(c, h) = 1$$
$$\det_1(c, h) = 2h$$
$$\det_2(c, h) = 2h(16h^2 + 2hc - 10h + c)$$

The following is a fundamental result.

**Theorem 2.13** (Kac, ’78).

$$\det_n(c, h) = C_n \prod_{r,s \in \mathbb{Z}_+} (h - h_{r,s}(c))^{p(n-rs)}$$
where \( C_n > 0 \) depends only on \( n \), and \( h_{r,s}(c) \) is given by

\[
h_{r,s}(c) = \frac{1}{48}((13 - c)(r^2 + s^2) - \sqrt{(c - 1)(c - 25)(r^2 - s^2) - 24rs + 2(c - 1)})
\]

### 2.3.3. Unitarity

We say that the irreducible representation \( V(c,h) \) is unitary if the induced hermitian form is positive definite. A necessary condition is clearly that \( \det_n(c,h) \geq 0 \) for all \( n \).

**Theorem 2.14.**

1. If \( V(c,h) \) is unitary, then \( c \geq 0, h \geq 0 \).
2. If \( c \geq 1, h \geq 0 \), then \( V(c,h) \) is unitary.
3. If \( c > 1, h > 0 \), then \( V(c,h) = M(c,h) \).

**Proof.** (1) If \( v = L_n|h\rangle \), then \( \langle v,v \rangle = 2nh + (c/12)(n^3 - n) \). For \( n \) large, we conclude \( c \geq 0 \). For \( n = 1 \), we conclude \( h \geq 0 \). For (2) and (3), first assume \( c > 1, h > 0 \). Since there are unitary examples in this region it suffices to prove that \( \det_n(c,h) \) is nonzero throughout for all \( n \). Notice that we have the following neat change of variables.

\[
c(m) = 1 - \frac{6}{m(m+1)} \quad \text{(2.17)}
\]

\[
h_{r,s}(m) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)} \quad \text{(2.18)}
\]

for \( 1 < c < 25 \), \( h_{r,s}(c) \) is complex by (2.16), so there are no real zeros. If \( c \geq 25 \), then clearly \( m(m+1) < 0 \), so \( -1 < m < 0 \). Then

\[
((m+1)r - ms)^2 = ((1 - |m|r + |m|s)^2 \geq 1,
\]

and so \( h_{r,s} \leq 0 \). The case \( c \geq 1 \) and \( h \geq 0 \) follows by continuity. \( \square \)

The following result shows that unitarity in the regime \( 0 \leq c < 1 \), this happens only for a discrete set.

**Theorem 2.15** (Friedan-Qiu-Shenker, ’85-86). If \( V(c,h) \) is unitary, \( 0 \leq c < 1 \), then \( (c,h) = (c(m),h_{r,s}(m)) \) for some \( m,r,s \in \mathbb{Z}_+ \), \( 1 \leq s \leq r < m \), where \( c(m) \) and \( h_{r,s}(m) \) are as in (2.17) and (2.18).

The collection \( \{c(m),h_{r,s}(m)\}, m = 2,3,\ldots \) is called the discrete series. Later on, a different parametrization of \( h_{r,s}(c) \) for \( 0 \leq c < 1 \) will be useful. Namely, it follows directly from the expression (2.16) that we can write

\[
h_{r,s}(c) = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2,
\]

\[
h_0 = \frac{1}{24}(c - 1)
\]

\[
\alpha_\pm = \sqrt{1 - c} \pm \sqrt{25 - c}
\]

\[
\sqrt{24}
\]

### 3. Conformal Fields

#### 3.1. 2d field theory.
3.1.1. **Definition.** Let $V$ be a hermitian vector space. We call the elements of $V$ states. Fields $\phi$ are formal distributions on $\mathbb{C}$ with values in $\text{End} V$. A distinguished field is the identity $\mathbf{I}$. We will always assume an expansion

\begin{equation}
\phi(z) = \sum_{n \in \mathbb{Z}} \phi^{(n)} z^{-n-1}
\end{equation}

where for any $v \in V$, $\phi^{(n)}(v) = 0$ for $n$ sufficiently large. The labeling is somewhat arbitrary. Define

\begin{equation}
\text{res}_z \phi(z) = \phi^{(0)}
\end{equation}

Then with the labeling above, $\text{res}_z \phi(z) z^n = \phi^{(n)}$.

By a formal distribution $a(z, w)$ we mean an $\text{End}(V)$ valued formal power series in monomials $z^m w^n$, $m, n \in \mathbb{Z}$, with each coefficient convergent. We say $a(z, w)$ is holomorphic in $z$, say, if all coefficients of $z^m w^n$ vanish for $m < 0$.

**Remark 3.1.** If $a(z, w)$ is a formal distribution then so is $p(z, w)a(z, w)$ for any polynomial $p(z, w)$. However, expressions such as $(z - w)^{-j} a(z, w)$ only become formal distributions once an expansion of $(z - w)^{-j}$ in terms of monomials is specified.

An important notion in field theory is locality. Namely, the fields at different points should commute with each other: $[\phi_1(z), \phi_2(w)] = 0$ if $z \neq w$. In terms of distributions this means that $[\phi_1(z), \phi_2(w)]$ is a sum of derivatives of delta functions $\delta(z - w)$. It is clearly desirable to axiomatize that such a sum be finite. Hence, we arrive at the following

**Definition 3.2.** We say that a collection of fields $\{\phi_\alpha\}$ is mutually local if for each $\alpha, \beta$

\begin{equation}
(z - w)^N [\phi_\alpha(z), \phi_\beta(w)] = 0
\end{equation}

for $N$ sufficiently large (depending on $\alpha, \beta$).

Locality imposes strong conditions on the fields. For example, $\phi_\alpha$ and $\phi_\beta$ are mutually local as above if and only if

\begin{equation}
\sum_{j=0}^{N} \binom{N}{j} [\phi_\alpha^{(m+j)}, \phi_\beta^{(n-j)}] = 0
\end{equation}

for all $m, n$.

**Definition 3.3.** A 2d field theory consists of

(1) a hermitian vector space $V$,

(2) a linear space of mutually local fields $\mathcal{LF}$,

(3) a linear map $\Phi : V \to \mathcal{LF}$,

(4) a vector $|0\rangle \in V$, and

(5) an element $\tau \in \text{End}(V)$,

satisfying the following properties.

(1) $\Phi(|0\rangle) = \mathbf{I}$;
(2) For all \( a \in V, \phi_a |0\rangle|_{z=0} = a \), where \( \phi_a = \Phi(a) \);

(3) For all \( \phi \in \mathcal{L}F \),

\[
\tau|0\rangle = 0
\]

\[
[\tau, \phi(z)] = \partial \phi(z)
\]

The map \( \Phi \) is sometimes called a \textit{state-field correspondence}. We will use the notation \( \phi_a = \Phi(a) \), and

\[
\phi_a = \sum_{k \in \mathbb{Z}} a^{(k)} z^{-k-1}
\]

Notice that the assumptions imply

\[
\tau a = \tau \phi_a |0\rangle|_{z=0} = [\tau, \phi_a]|0\rangle|_{z=0} = \partial \phi_a |0\rangle|_{z=0} = a^{(-2)} |0\rangle,
\]

and then by induction we have

\[
\tau^k a = k! a^{(-k-1)} |0\rangle
\]

\[
e^{z\tau} a = \phi_a(z)|0\rangle
\]

Conversely, if \( \phi(z) \in \mathcal{L}F \) such that \( \phi(z)|0\rangle \) is holomorphic, then defining \( a = \phi(z)|0\rangle|_{z=0} \) it follows from condition (3) that \( \phi(z)|0\rangle = e^{z\tau} a \).

3.1.2. \textit{Correlation functions and ordering.} In QFT the ordering of products is important. This is normally done with respect to time. Identifying time with the radius, time ordering becomes equivalent to radial ordering. Given distinct points \( z_1, \ldots, z_n \) and operators \( \phi_1(z), \ldots, \phi_n(z) \), we define the time (or radially) ordered product

\[
\mathcal{T}(\phi_1(z_1) \ldots \phi_n(z_n)) = \phi_{\sigma(1)}(z_{\sigma(1)}) \cdots \phi_{\sigma(n)}(z_{\sigma(n)})
\]

where \( \sigma \) is the unique permutation such that

\[
|z_{\sigma(1)}| > |z_{\sigma(2)}| > \cdots > |z_{\sigma(n)}|
\]

\textbf{Definition 3.4.} Let \( \{\phi_i(z)\}_{i=1}^n \) be fields in a 2d field theory. Correlation functions are defined as follows: choose distinct points \( z_1, \ldots, z_n \). Then we set

\[
\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle = \langle 0| \mathcal{T}(\phi_1(z_1) \cdots \phi_n(z_n))|0\rangle
\]

\textit{Solving} a given a 2d field theory means computing all correlation functions. We shall sometimes drop the time ordering notation. Whenever we are thinking of products in terms of correlation functions, the time ordering is assumed.

We also define another ordering scheme. Given a field (3.1), let

\[
\phi^+(z) = \sum_{n \leq -1} \phi^{(n)} z^{-n-1}
\]

\[
\phi^-(z) = \sum_{n \geq 0} \phi^{(n)} z^{-n-1}
\]
Definition 3.5. The normal ordering of two fields is defined by

\[ \phi_1(z) \phi_2(w) := \phi_1^+(z) \phi_2(w) + \phi_2(w) \phi_1^-(z) . \]

Note the following:

\begin{align*}
\phi_1(z) \phi_2(w) &= [\phi_1^-(z), \phi_2(w)] + \phi_1(z) \phi_2(w) \tag{3.5} \\
\phi_2(w) \phi_1(z) &= -[\phi_1^+(z), \phi_2(w)] + \phi_1(z) \phi_2(w) \tag{3.6}
\end{align*}

It is easy to see that the normally ordered product is regular as \( w \to z \), so that we have a well-defined field : \( \phi_1(z) \phi_2(z) : \). We will also see several properties of this product.

- it is not, in general, associative;
- it is commutative (up to sign) for free fields;
- it preserves locality.

3.2. Operator product expansions.

3.2.1. Formal Dirac distribution. We define the formal Dirac distribution as follows

\[ \delta(z, w) = 1 + \sum_{n \in \mathbb{Z}} \left( \frac{w}{z} \right)^n . \tag{3.7} \]

Lemma 3.6. The distribution \( \delta(z, w) \) satisfies

1. \( \delta(z, w) = \delta(w, z) \);
2. For any field \( \phi(z) \), \( \phi(z) \delta(z, w) \) is a formal distribution;
3. \( \partial_z \delta(z, w) = -\partial_w \delta(z, w) \);
4. \( (z - w) \partial_w^{j+1} \delta(z, w) = (j + 1) \partial_w^j \delta(z, w) \);
5. \( (z - w)^{j+1} \partial_z^j \delta(z, w) = 0 \).

Proof. These are mostly just computations. Perhaps only (2) is a bit mysterious. To see this, note that

\[ \phi(z) \delta(z, w) = \sum_{m, n \in \mathbb{Z}} \phi^{(m+n)} z^{-m-1} w^{-n-1} . \]

In particular, each coefficient in the expansion is either \( z \) or \( w \) is a field in the other variable. \( \Box \)

We introduce some more notation. The symbol \( \iota_{z,w} \) (resp. \( \iota_{w,z} \)) will denote the power series expansion in the region \( |z| > |w| \) (resp. \( |w| > |z| \)). Then we have the following

Lemma 3.7. As formal distributions,

\[ \frac{1}{j!} \partial_w^j \delta(z, w) = \iota_{z,w} \frac{1}{(z - w)^{j+1}} - \iota_{w,z} \frac{1}{(z - w)^{j+1}} . \]
Proof. Recall that
\[
\frac{1}{(1-x)^{j+1}} = \sum_{m=0}^{\infty} \binom{m+j}{j} x^m
\]
for \(|x| < 1\). Hence,
\[
i_{z,w} \frac{1}{(z-w)^{j+1}} = \sum_{m=0}^{\infty} \binom{m+j}{j} z^{-m-j-1} w^m
\]
(3.8)
\[
i_{w,z} \frac{1}{(z-w)^{j+1}} = (-1)^{j+1} \sum_{m=0}^{\infty} \binom{m+j}{j} z^{m-w} w^{-m-j-1}
\]
(3.9)

Now
\[
\frac{\partial_j^\delta(z,w)}{z} = \sum_{n \geq j} \frac{m(m-1) \cdots (m-j+1) w^{m-j}}{z^m} + \sum_{m < 0} m(m-1) \cdots (m-j+1) z^{-m-1} w^{m-j}
\]
In the first substitute \(n = m - j\), in the second substitute \(n = -m - 1\). Then
\[
\frac{1}{j!} \frac{\partial_j^\delta(z,w)}{z} = \sum_{n=0}^{\infty} \frac{(n+j) \cdots (n+1) z^{-n-j-1} w^n}{z^m} + \sum_{n=0}^{\infty} \frac{(-n-1)(-n-2) \cdots (-n-j) z^n w^{-n-j-1}}{z^m}
\]
\[
= i_{z,w} \frac{1}{(z-w)^{j+1}} - i_{w,z} \frac{1}{(z-w)^{j+1}}
\]
by (3.8) and (3.9). \(\square\)

3.2.2. The main result. The main goal of this section is to prove the following

**Theorem 3.8** (OPE Expansion). Suppose \(\phi_1(z)\) and \(\phi_2(w)\) are mutually local fields. Then there is an expansion
\[
\mathcal{F}(\phi_1(z)\phi_2(w)) = \sum_{j=0}^{N-1} \frac{c_j(w)}{(z-w)^{j+1}} + :\phi_1(z)\phi_2(w) :,
\]
where \(c_j(w) = \text{res}_z[\phi_1(z), \phi_2(w)](z-w)^j\) where the normally ordered term \(\phi_1(z)\phi_2(w) :\) is defined in Definition 3.5. The formal distribution in the sum on the right hand side is defined using \(i_{z,w}(z-w)^{-j-1}\) or \(i_{w,z}(z-w)^{-j-1}\) in accordance with the radial ordering.
It is very common in the literature to drop the time ordering notation. One also typically drops the regular part of the OPE. Hence, the result above is often written
\[
\phi_1(z)\phi_2(w) \sim \sum_{j=0}^{N-1} \frac{c_j(w)}{(z-w)^{j+1}}.
\]
We note that it is \textit{not obvious} that the fields \(c_j\) are mutually local with \(\phi_1\) and \(\phi_2\). This will be proved in Theorem 3.17 below.

The \(c_j\)'s are easily computed given in terms of commutators in mode expansions. For example, if
\[
\phi_1(z) = \sum_{m \in \mathbb{Z}} \phi_1^m z^{-m-1}, \phi_2(w) = \sum_{n \in \mathbb{Z}} \phi_2^n w^{-n-1},
\]
then one easily computes
\[
\begin{align*}
\text{(3.11)} \quad & c_j(w) = \sum_{n \in \mathbb{Z}} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} [\phi_1^i, \phi_2^{n+j-i}] w^{-n-1} \\
\text{(3.12)} \quad & c_j^n = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} [\phi_1^i, \phi_2^{n+j-i}]
\end{align*}
\]

Intuitively, the condition of locality means that the commutator, as a distribution, is a finite sum of (\(\text{End}(V)\)-valued) derivatives of the Dirac function along the diagonal. The role of the Dirac function is played by \(\delta(z,w)\) defined in (3.7). This is justified by the following key result.

\textbf{Lemma 3.9.} Suppose \(a(z,w)\) is a formal distribution such that \(\text{res}_z a(z,w)(z-w)^j = 0\) for some \(j \geq N\). Then
\[
a(z,w) = \sum_{j=0}^{N-1} c_j(w) \frac{1}{j!} \partial_w^j \delta(z,w) + b(z,w)
\]
where \(c_j(w) = \text{res}_z a(z,w)(z-w)^j\), and \(b(z,w)\) is holomorphic in \(z\).

\textit{Proof.} Note that by induction on Lemma 3.6 (4),
\[
\text{res}_z (z-w)^k \partial_w^j \delta(z,w) = \begin{cases} 0 & k \neq j \\ j! & k = j \end{cases}
\]
Hence, defining \(b(z,w)\) as above, write
\[
b(z,w) = \sum_{n \in \mathbb{Z}} b_n(w) z^n.
\]
Then \(\text{res}_z b(z,w) = 0\) implies \(b_{-1}(w) = 0\). \(\text{res}_z b(z,w)(z-w) = 0\) implies \(b_{-2}(w) = b_{-1}(w) w = 0\), and so on. \(\Box\)
Corollary 3.10. Suppose \( \phi_1(z) \) and \( \phi_2(w) \) are mutually local fields. Then there is an expansion

\[
[\phi_1(z), \phi_2(w)] = \sum_{j=0}^{N-1} c_j(w) \frac{1}{j!} \partial^j_w \delta(z,w)
\]

**Proof.** In the expansion from Lemma 3.9 we have \( b(z,w)(z-w)^N = 0 \), by Lemma 3.6 (5). But \( b(z,w) \) is holomorphic in \( z \), so it vanishes. \( \square \)

Since

\[
[\phi_1(z), \phi_2(w)] = [\phi_1^+(z), \phi_2(w)] + [\phi_1^-(z), \phi_2(w)]
\]

then by comparing positive and negative coefficients of \( z \) in Lemma 3.9 with the result of Lemma 3.7 we have

**Lemma 3.11.**

\[
[\phi_1^-(z), \phi_2(w)] = \sum_{j=0}^{N-1} c_j(w) \frac{1}{(z-w)^{j+1}}
\]

\[
- [\phi_1^+(z), \phi_2(w)] = \sum_{j=0}^{N-1} c_j(w) \frac{1}{(z-w)^{j+1}}
\]

**Proof of Theorem 3.8.** The result now follows from (3.5) and (3.6) combined with Lemma 3.11. \( \square \)

3.2.3. Commutators and OPE’s. Given an OPE expansion, one can derive commutators of mode expansions. The following can be thought of as an inversion of (3.13).

**Theorem 3.12.** Let \( \phi_1(z) \) and \( \phi_2(w) \) be local fields with mode expansions (3.11). Let \( c_j^n \) be the mode expansions in the OPE of \( \phi_1(z)\phi_2(w) \). Then

\[
[\phi_1^m, \phi_2^n] = \sum_{j=0}^{N-1} \binom{m}{j} c_j^{m+n-j}.
\]

**Proof.** This is a consequence of the following well-known picture in conformal field theory.
Here, $C, C_1, C_2$ are contours about the origin, and $C_w$ is a contour about $w$ not containing the origin. Then

$$[\phi^m_1, \phi^n_2] = \frac{1}{2\pi i} \int_C dw \frac{1}{2\pi i} \int_C dz z^m [\phi_1(z), \phi_2(w)]$$

$$= \frac{1}{2\pi i} \int_C dw \frac{1}{2\pi i} \int_{C_2-C_1} dz z^m \mathcal{T}(\phi_1(z)\phi_2(w))$$

$$= \frac{1}{2\pi i} \int_C dw \frac{1}{2\pi i} \int_{C_2-C_1} dz z^m \sum_{j=0}^{N-1} \frac{c_j(w)}{(z-w)^{j+1}}$$

$$= \frac{1}{2\pi i} \int_C dw \frac{1}{2\pi i} \int_{C_w} dz z^m \sum_{j=0}^{N-1} \frac{c_j(w)}{(z-w)^{j+1}}$$

$$= \frac{1}{2\pi i} \int_C dw \sum_{j=0}^{N-1} \left( \binom{m}{j} \right) c_j(w) w^{m+n-j}$$

$$= \frac{1}{2\pi i} \int_C dw \sum_{j=0}^{N-1} \left( \binom{m}{j} \right) c_j^k w^{m+n-j-k-1}$$

$$= \sum_{j=0}^{N-1} \left( \binom{m}{j} \right) c_j^{m+n-j}.$$  

\[ \square \]

3.2.4. Examples. Free bosons. As a first example, consider the field

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$$

where \( \{a_n\} \) satisfy the commutation relations (2.4). Then in this case

$$c_j^n = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} [a_i, a_{n+j-i}] = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i\delta_{n+j,0} = \frac{d}{dx} (x-1)^j \big|_{x=1} \delta_{n+j,0} = \delta_{j,1} \delta_{n+j,0}$$
Conversely, given these values for $c_j^n$, the commutation relations (2.4) follows from Theorem 3.12. Hence, we have shown

**Lemma 3.13.** We have the following OPE,

$$a(z)a(w) \sim \frac{1}{(z-w)^2}.$$  
(3.15)

Conversely, given the above OPE, the commutation relations (2.4) are satisfied.

**Free fermions.** As a second example, consider the fermion field

$$\psi(z) = \sum_{n \in 1/2 + \mathbb{Z}} \psi_n z^{-n-1/2}$$  
(3.16)

where $\{\psi_n\}$ satisfy the anti-commutation relations (2.7) (this is the NS sector). Then in this case

$$c_j^n = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \{\psi_i, \psi_{n+j-i}\} = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \delta_{n+j,0} = \delta_{j,0} \delta_{n+j,0}$$

Hence, we have shown

**Lemma 3.14.** We have the following OPE:

$$\psi(z)\psi(w) \sim \frac{1}{z-w}.$$  
(3.17)

Conversely, given the above OPE, the anti-commutation relations (2.7) are satisfied.

**The bc ghost system.** Consider again the bc-ghost system (2.11). Write the mode expansion

$$b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-\lambda}$$  
(3.18)
$$c(z) = \sum_{n \in \mathbb{Z}} c_n z^{-n-1+\lambda}$$  
(3.19)

**Lemma 3.15.** We have the following OPE:

$$b(z)c(w) \sim \frac{1}{z-w}.$$  
(3.20)

Conversely, given the above OPE, the anti-commutation relations (2.11) are satisfied.

**Energy-Momentum tensor.** As a last example, consider the field

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

where the $\{L_n\}$ satisfy the Virasoro conditions (2.3). Then

**Lemma 3.16.** We have the following OPE:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$  
(3.21)

Conversely, given the above OPE, the commutation relations (2.3) are satisfied.
Proof. By (3.13),
\[ c_j^p = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} [L_{i-1}, L_{n+j-i-1}] \]
\[ = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \left\{ (2i - n - j)L_{n+j-2} + \frac{c}{12} (i - 1)(i - 2)\delta_{n+j-2,0} \right\} \]

The sum in the last term is the same as \((c/12)\delta_{n+j-2,0}\) times
\[ \left( \frac{d}{dx} \right)^3 (x - 1)^j \right|_{x=1} = \begin{cases} 6 & j = 3 \\ 0 & j \neq 3 \end{cases} \]

Similarly,
\[ \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} 2i = 2 \frac{d}{dx} (x - 1)^j \right|_{x=1} = \begin{cases} 2 & j = 1 \\ 0 & j \neq 1 \end{cases} \]
\[ \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (n + j) = \begin{cases} n & j = 0 \\ 0 & j \neq 0 \end{cases} \]

\[ \square \]

3.2.5. Closedness of the operator algebra. It is an important property of the operator algebra that condition of locality is closed with respect to the operator product expansion. We prove this in the present section. In fact, there is the stronger result.

**Theorem 3.17.** In a 2d field theory, the set of fields \( \{ \phi_a \}_{a \in V} \) is closed under operator product expansion. Moreover, if \( a, b \in V \), then the coefficients of the OPE, \( \phi_a(z)\phi_b(w) \) are given by

(3.22)
\[ c_j(w) = \phi_{a(j)b}(w) \] .

This remarkable result means that the fields in a 2d field theory form an algebra. The determination of which fields appear in the OPE of two given fields is called a **fusion rule**. We will say much more about these later on.

**Lemma 3.18** (Dong’s Lemma). Given mutually local fields \( \phi_1(z), \phi_2(z) \), then if \( c_j(z) \) is one of the fields appearing in the OPE (3.10), the set \( \{ \phi_1(z), \phi_2(z), c_j(z) : \phi_1(z)\phi_2(z) : \} \) is mutually local.

The following is the converse to (3.3).

**Lemma 3.19** (Uniqueness Lemma). Let \( \phi(z) \) be a field which is local with respect to the fields in a 2d field theory. Moreover, suppose that for some \( a \in V \),
\[ \phi(z)|0\rangle = e^{z\tau} a . \]

Then \( \phi(z) = \phi_a(z) \).
Proof. Let $b \in V$. Then for $N$ sufficiently large, and using (3.3),

\[
(z - w)^N \phi(z) \phi_b(w) |0\rangle = (z - w)^N \phi_b(w) \phi(z) |0\rangle
\]

\[
(z - w)^N \phi(z) e^{w t} b = (z - w)^N \phi_b(w) e^{z t} a
\]

\[
(z - w)^N \phi(z) e^{w t} b = (z - w)^N \phi_b(w) \phi_a(z) |0\rangle
\]

\[
(z - w)^N \phi(z) e^{w t} b = (z - w)^N \phi_a(z) \phi_b(w) |0\rangle
\]

\[
(z - w)^N \phi(z) e^{w t} b = (z - w)^N \phi_a(z) e^{w t} b |0\rangle
\]

Now set $w = 0$ and divide by $z^N$ to conclude $\phi(z)b = \phi_a(z)b$ for all $b \in V$. \(\square\)

Corollary 3.20. Given $a \in V$, $\phi_{\tau}^k a(z) = \partial^k \phi_a(z)$.

Proof. Immediate from (3.3) and Lemma 3.19. \(\square\)

Corollary 3.21. Given $a,b \in V$, $\phi_a(z) \phi_b(z) := \phi_c(z)$, where $c = a(-1)b$.

Proof. It is simple to check that

\[
[\tau, : \phi_a(z) \phi_b(z) :] = \partial : \phi_a(z) \phi_b(z) :
\]

\[
: \phi_a(z) \phi_b(z) : |0\rangle \big|_{z=0} = a(-1)b
\]

It follows that

\[
: \phi_a(z) \phi_b(z) : |0\rangle = e^{z t} a(-1)b
\]

Since by Lemma 3.18 the normal ordering is local, the result follows from Lemma 3.19. \(\square\)

In the same way, one proves

Corollary 3.22. In the 2d field theory, the image of $\Phi$ consists precisely of those $\phi \in \mathcal{LF}$ such that $\phi(z) |0\rangle$ is holomorphic.

Given this, it may be more natural to assume from the beginning that fields have asymptotic states, in which case the state-field correspondence is an isomorphism.

Proof of Theorem 3.17. We first claim that

\[
\partial c_j(z) |0\rangle = \tau c_j(z) |0\rangle .
\]

Since the same equation holds for $\phi_{a(j)b}(z)$, and by (3.13),

\[
c_j(z) |0\rangle \big|_{z=0} = c_j^{(-1)} |0\rangle
\]

\[
= \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} [\phi_a^i, \phi_b^{j-i-1}] |0\rangle
\]

\[
= \phi_a^i \phi_b^{(-1)} |0\rangle
\]

\[
= \phi_{a(j)(b)}(z) |0\rangle \big|_{z=0}
\]
we see that the claim implies (3.22). Now the claim follows if we show \([\tau, c^m_n] = -nc^m_{n-1}\), for all \(n \in \mathbb{Z}\). But again using (3.13),

\[
[\tau, c^m_n] = \sum_{i=0}^{j} (-1)^{j-i} {j \choose i} [\tau, [\phi^i_a, \phi^{n+j-i}_b]]
\]

\[
= -\sum_{i=0}^{j} (-1)^{j-i} {j \choose i} \{ [\phi^i_a, [\phi^{n+j-i}_b, \tau]] + [\phi^{n+j-i}_b, [\tau, \phi^i_a]] \}
\]

\[
= \sum_{i=0}^{j} (-1)^{j-i} {j \choose i} \{ [\phi^i_a, [\tau, \phi^{n+j-i}_b]] + [[\tau, \phi^i_a], \phi^{n+j-i}_b] \}
\]

\[
= -\sum_{i=0}^{j} (-1)^{j-i} {j \choose i} \{ (n+j-i)[\phi^{i+j-i}_a, \phi^{n+j-i}_b] + i[\phi^{i+j-i}_a, \phi^{n+j-i}_b] \}
\]

\[
= -\sum_{i=0}^{j} (-1)^{j-i} {j \choose i} \{ (n+j-i)\left( {j \choose i} - (i+1)\left( {j \choose i+1} \right) \right) [\phi^{i+j-i}_a, \phi^{n+j-i}_b] \}
\]

\[
= -n\sum_{i=0}^{j} (-1)^{j-i} {j \choose i} [\phi^{i+j-i}_a, \phi^{n+j-i}_b] = -nc^{n-1}_{j-1}
\]

\[\square\]

3.3. Definition of conformal field theory.

3.3.1. Energy-momentum tensor. Suppose a 2d field theory contains a local field \(T(z) = \phi_v(z)\) satisfying the OPE (3.21). Given the mode expansion

\[T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}\]

the OPE is equivalent to saying the \(\{L_n\}\) satisfy (2.3). Such a field will be called an Energy-Momentum tensor (also called Stress-Energy tensor in the literature). Notice that

\[v = L_{-2}|0\rangle , \quad L_m|0\rangle = 0 , \quad m \geq -1 .\]

In particular, the vacuum is invariant under the \(\mathfrak{sl}(2,\mathbb{C})\) subalgebra.

**Definition 3.23.** A conformal field theory is a 2d-field theory with vector \(v \in \mathcal{V}\) such that \(T(z) = \phi_v(z)\) satisfies

1. (3.21);
2. \(L_0\) is diagonalizable on \(\mathcal{V}\);
3. \(L_{-1} = \tau\).
3.3.2. Primary fields.

**Definition 3.24.** A field \( \phi_a \) is called a primary field if its OPE with the energy-momentum tensor is of the form

\[
T(z)\phi_a(w) \sim \frac{\Delta_a \phi_a(w)}{(z-w)^2} + \frac{\partial \phi_a(w)}{z-w}
\]

(3.24)

Note that primary fields are quasi-primary and \( \Delta_a \) is the conformal weight. Also, \( T(z) \) itself is quasiprimary but not primary (unless \( c = 0 \)). This is why the central charge is sometimes called the conformal anomaly: the energy-momentum tensor is holomorphic but not really a tensor (or; if one requires \( T \) to be tensorial then it won’t be holomorphic).

**Proposition 3.25.** A field \( \phi_a \) is primary with weight \( \Delta_a \) if and only if any one of the following equivalent conditions holds.

1. \( L_m a = 0 \) for \( m > 0 \) and \( L_0 a = \Delta_a a \);
2. \( [L_m, \phi_a(z)] = z^m(z \partial z + (m+1)\Delta_a)\phi_a(z) \);
3. \( [L_m, a^{(n)}] = ((m+1)\Delta_a - (m+n+1))a^{(m+n)} \), for all \( m,n \).

**Proof.** Clearly, (2) \( \iff \) (3). (3) \( \iff \) (3.24) by (3.13) and Theorem 3.12. (3) \( \Rightarrow \) (1), since \( a = \phi_a^{-1}\langle 0 \rangle \), and by (3.23)

\[
L_m a = [L_m, \phi_a^{-1}]\langle 0 \rangle = ((m+1)\Delta_a - m)\phi_a^{(m-1)}\langle 0 \rangle 0
\]

if \( m \geq 1 \). To show (1) \( \Rightarrow \) (3.24), from (3.22) we have \( c_0(w) = \phi_{L_{-1}}(w) \), \( c_1(w) = \phi_{L_0}(w) \), and \( c_j(w) \equiv 0 \) for \( j \geq 2 \). Hence, \( c_1(w) = \Delta_a \phi_a(w) \), and by the fact that \( L_{-1} = \tau \) and Corollary 3.20, \( c_0(w) = \partial \phi_a(w) \). \( \square \)

From Proposition 3.25 we see that primary fields are in 1-1 correspondence with highest vectors for the Virasoro representation \( V \). We note that for a field of weight \( \Delta_a \) is common to write the mode expansion

(3.25)

\[
\phi_a(z) = \sum_{n \in \mathbb{Z}} a^{(n)} z^{-n-\Delta}
\]

Then condition (3) in Proposition 3.25 becomes

(3.26)

\[
[L_m, a^{(n)}] = (m(\Delta - 1) - n)a^{(m+n)}
\]

**Definition 3.26.** A field \( \phi_a \) is called an eigendistribution with weight \( \Delta_a \) if (3) (or equivalently, (2)) Definition 3.24 holds for \( m = 0 \).

When we say a field has weight \( \Delta \), we implicitly mean that it is an eigendistribution. The generator \( L_0 \) corresponds to scaling, so the eigendistributions are the fields with a well-defined scaling law. To make this precise, note the following
Proposition 3.27. Let $\phi_1, \ldots, \phi_N$ have conformal weights $\Delta_1, \ldots, \Delta_N$, with $\Delta = \sum_{i=1}^N \Delta_i$. Then

$$\langle \phi_1(\lambda z_1) \cdots \phi_N(\lambda z_N) \rangle = \lambda^{-\Delta} \langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle$$

Proof. We have

$$\frac{d}{d\lambda} \langle \phi_1(\lambda z_1) \cdots \phi_N(\lambda z_N) \rangle = \sum_{i=1}^N \frac{1}{\lambda} \langle \phi_1(\lambda z_1) \cdots \lambda z_i \partial \phi_i(\lambda z_i) \cdots \phi_N(\lambda z_N) \rangle$$

$$= \sum_{i=1}^N \frac{-\Delta_i}{\lambda} \langle \phi_1(\lambda z_1) \cdots \phi_N(\lambda z_N) \rangle$$

$$= \sum_{i=1}^N \frac{1}{\lambda} \langle \phi_1(\lambda z_1) \cdots [L_0, \phi_i(\lambda z_i)] \cdots \phi_N(\lambda z_N) \rangle$$

$$= \frac{d}{d\lambda} \log \lambda^{-\Delta} \langle \phi_1(\lambda z_1) \cdots \phi_N(\lambda z_N) \rangle - \frac{1}{\lambda} \langle [L_0, \phi_1(\lambda z_1) \cdots \phi_N(\lambda z_N)] \rangle.$$ 

The second term vanishes, since $L_0|0\rangle = 0$. The rest follows. \qed

We have the following obvious

Proposition 3.28. If $\phi_1$, $\phi_2$ have weight $\Delta_1$, $\Delta_2$, then

1. $\partial \phi_1$ has weight $\Delta_1 + 1$;
2. $\partial \phi_1(z) \phi_2(z)$ has weight $\Delta_1 + \Delta_2$;
3. The coefficient $c_j$ in the OPE $\phi_1 \phi_2$ has weight $\Delta_1 + \Delta_2 - j - 1$.

Definition 3.29. A field $\phi_a$ of weight $\Delta_a$ is called quasiprimary if condition (3) (or equivalently, (2)) in Definition 3.24 holds for $L_m$, $m = 0, \pm 1$.

Condition (2) in Definition 3.24 may be regarded as the invariance of $\phi$ regarded as a section of the bundle $(T^*\mathbb{C})^\Delta \otimes \text{End}(V)$.

3.3.3. Free bosons. Consider again the field $a(z)$ defined in (3.14). Then if we define

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = -(1/2): a(z) a(z) :$$

the $\{L_n\}$ satisfy the Virasoro condition with $c = 1$ (see Proposition 2.5). Moreover, one computes

$$T(z) \phi_a(w) \sim \frac{a(w)}{(z-w)^2} + \frac{\partial a(w)}{z-w}$$

so $a(z)$ is a primary field of weight 1.
3.3.4. Free fermions. If \( \psi(z) \) is defined as in (3.16), then if we define

\[
T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = -(1/2) : \psi(z) \partial \psi(z) :
\]

the \( \{L_n\} \) satisfy the Virasoro condition with \( c = 1/2 \) (see Proposition 2.8). Moreover, one computes

\[
T(z) \phi_\lambda(w) \sim \frac{(1/2) \psi(w)}{(z-w)^2} + \frac{\partial \psi(w)}{z-w}
\]

so \( \psi(z) \) is a primary field of weight 1/2.

3.3.5. bc ghost system. In terms of the fields (3.18) and (3.19), the Energy-Momentum tensor is

\[
(3.27) \quad T(z) =: (1 - \lambda) \partial b(z) c(z) - \lambda b(z) \partial c(z) :
\]

**Lemma 3.30.** In terms of the mode expansion \( T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \),

\[ L_n = \sum_{j \in \mathbb{Z}} (\lambda n - j) : b_j c_{n-j} : \]

**Proof.** We have

\[
\partial b^+(z) = - \sum_{n \leq -\lambda} (n + \lambda) b_n z^{-n-\lambda-1}
\]

\[
\partial b^-(z) = - \sum_{n \geq -\lambda} (n + \lambda) b_n z^{-n-\lambda-1}
\]

\[
\partial c(z) = - \sum_{n \in \mathbb{Z}} (n + 1 - \lambda) c_n z^{-n-2+\lambda}
\]

Then from Definition 3.5,

\[
: \partial b(z) c(z) : = \partial b^+(z) c(z) - c(z) \partial b^-(z)
\]

\[
= - \sum_{n \in \mathbb{Z}} \sum_{m \leq -\lambda} (m + \lambda) b_m c_n z^{-m-n-2} + \sum_{n \in \mathbb{Z}} \sum_{m \geq -\lambda + 1} (m + \lambda) c_n b_m z^{-m-n-2}
\]

\[
: b(z) \partial c(z) : = b^+(z) \partial c(z) - \partial c(z) b^-(z)
\]

\[
= - \sum_{n \in \mathbb{Z}} \sum_{m \leq -\lambda} (n + 1 - \lambda) b_m c_n z^{-m-n-2} + \sum_{n \in \mathbb{Z}} \sum_{m \geq -\lambda + 1} (n + 1 - \lambda) c_n b_m z^{-m-n-2}
\]
So
\[T(z) = \sum_{n \in \mathbb{Z}} (\lambda(n + 1 - \lambda) - (1 - \lambda)(m + \lambda)) b_m c_n z^{-m-n-2}
- \sum_{n \in \mathbb{Z}} (\lambda(n + 1 - \lambda) - (1 - \lambda)(m + \lambda)) c_n b_m z^{-m-n-2}
= \sum_{n \in \mathbb{Z}} (\lambda(m + n) - m) b_m c_n z^{-m-n-2}
- \sum_{n \in \mathbb{Z}} (\lambda(m + n) - m) c_n b_m z^{-m-n-2}\]

The result now follows by relabelling and using the normal ordering in (2.12). □

It follows from Proposition 2.9 that \(\{L_n\}\) satisfy the Virasoro condition with 
\[c = -2(6\lambda^2 - 6\lambda + 1)\]

**Lemma 3.31.** The fields \(b(z), c(z)\) are primary of weight \(\lambda, 1 - \lambda\), respectively.

**Proof.** By (3.26) we need to show
\[[L_m, b_n] = (m(\lambda - 1) - n)b_{m+n}\]

But
\[[L_m, b_n] = \sum_{j \leq -\lambda} (\lambda m - j) (b_j c_{m-j} b_n - b_n b_j c_{m-j})
- \sum_{j \geq -\lambda + 1} (\lambda m - j) (c_{m-j} b_j b_n - b_n c_{m-j} b_j)
= \sum_{j \in \mathbb{Z}} (\lambda m - j) \{b_n, c_{m-j}\} b_j
= \sum_{j \in \mathbb{Z}} (\lambda m - j) \delta_{m+n-j,0} b_j
= (\lambda m - (m + n)) b_{m+n}.\]

The computation for \(c(z)\) is similar. □

We also point out that \(T(z)\) comes from a lagrangian formulation.

**3.3.6. Ward identities.** The Energy-Momentum tensor is the generator of conformal transformations. To see this, we first state

**Theorem 3.32** (Conformal Ward Identities). Let \(\phi_1, \ldots, \phi_N\) be quasi-primary fields with dimensions \(\Delta_1, \ldots, \Delta_N\). Then

1. \(\sum_{i=1}^{N} \partial_\zeta \langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle = 0.\)
2. \(\sum_{i=1}^{N} (\zeta_i \partial_\zeta + \Delta_i) \langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle = 0.\)
3. \(\sum_{i=1}^{N} (\zeta_i^2 \partial_\zeta + 2\zeta_i \Delta_i) \langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle = 0.\)
Proof. Note that by the definition of quasi-primary, these identities are equivalent to
\[ \sum_{i=1}^{N} \langle \phi_1(z_1) \cdots [L_m, \phi_i(z_i)] \cdots \phi_N(z_N) \rangle = 0 , \]
for \( m = -1, 0, 1 \), respectively. But the left hand side above is
\[ \langle 0 [L_m, \phi_1(z_1) \cdots \phi_N(z_N)] 0 \rangle \]
and this vanishes, since \( L_m |0\rangle = 0 \) for \( m \geq -1 \) and \( L^*_m = L_{-m} \).
\( \square \)

**Theorem 3.33** (Second Ward Identities). Let \( \phi_1, \ldots, \phi_N \) be primary fields with dimensions \( \Delta_1, \ldots, \Delta_N \). Then
\[ \langle T(z) \phi_1(z_1) \cdots \phi_N(z_N) \rangle = \sum_{i=1}^{N} \left( \frac{\Delta_i}{(z-z_i)^2} + \frac{1}{(z-z_i)} \frac{\partial}{\partial z_i} \right) \langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle \]
for \(|z| > |z_1| > \cdots > |z_N|\).

Proof. Since \( L_m |0\rangle = 0 \) for \( m \geq -1 \), we have
\[ \langle T(z) \phi_1(z_1) \cdots \phi_N(z_N) \rangle = \sum_{m \geq 0} \langle L_m \phi_1(z_1) \cdots \phi_N(z_N) \rangle z^{-m-2} \]
\[ = \sum_{m \geq 0} \langle [L_m, \phi_1(z_1)] \cdots \phi_N(z_N) \rangle z^{-m-2} \]
\[ = \sum_{i=1}^{N} \sum_{m \geq 0} \langle \phi_1(z_1) \cdots [L_m, \phi_i(z_i)] \cdots \phi_N(z_N) \rangle z^{-m-2} \]
\[ = \sum_{i=1}^{N} \sum_{m \geq 0} (z_i^{m+1} \partial_{z_i} + \Delta_i(m+1)z_i^m) \langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle z^{-m-2} \]
\[ = \sum_{i=1}^{N} \sum_{m \geq 1} z_i^m z^{-m-1} \partial_{z_i} \langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle \]
\[ + \sum_{i=1}^{N} \sum_{m \geq 0} z_i^m z^{-m-2} \Delta_i(m+1) \langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle \]
Now by Theorem 3.32 (1), we can allow \( m = 0 \) in the first sum. Then the result follows from the identities
\[ \sum_{m=0}^{\infty} z_i^m z^{-m-1} = \frac{1}{z-z_i} \]
\[ \sum_{m=0}^{\infty} (m+1) z_i^m z^{-m-2} = \frac{1}{(z-z_i)^2} \]
\( \square \)
Let $\varepsilon(z)$ be an infinitesimal conformal transformation. Then
\[
\frac{\delta}{\delta \varepsilon} \langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle = \frac{1}{2\pi i} \int_C dz \varepsilon(z) \langle T(z) \phi_1(z_1) \cdots \phi_N(z_N) \rangle
\]
where $C$ is a contour containing all the points $z_i$.

Let us draw some consequences. First, by Theorem 3.32, it follows that for quasi-primary fields
\[
\langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle \prod_{i=1}^N (dz_i)^{\Delta_i}
\]
is an $SL(2, \mathbb{C})$-invariant form.

**Proposition 3.34.** If $\phi_1, \phi_2$ are quasi-primary of weights $\Delta_1, \Delta_2$, then
\[
\langle \phi_1(z_1) \phi_2(z_2) \rangle = \begin{cases} 
C & \Delta_1 = \Delta_2 = \Delta \\
(z_1 - z_2)^{2\Delta} & \Delta_1 \neq \Delta_2 
\end{cases}
\]

**Proof.** Set $f(z_1, z_2) = \langle \phi_1(z_1) \phi_2(z_2) \rangle$. Then from Theorem 3.32 (2) we have
\[
z_1^2 \partial_{z_1} f + z_1 z_2 \partial_{z_2} f + z_1 (\Delta_1 + \Delta_2) f = 0
\]
\[
z_1 z_2 \partial_{z_1} f + z_2^2 \partial_{z_2} f + z_2 (\Delta_1 + \Delta_2) f = 0
\]
Add them and use Theorem 3.32 (1) to get
\[
z_1^2 \partial_{z_1} f + z_2^2 \partial_{z_2} f + (z_1 + z_2)(\Delta_1 + \Delta_2) f = 0
\]
Substitute Theorem 3.32 (3) to find
\[
(z_2 - z_1)(\Delta_1 - \Delta_2) f = 0
\]
Hence, either $f \equiv 0$ or $\Delta_1 = \Delta_2$. Now use Theorem 3.32 (1) and (2) to write
\[
(z_1 - z_2)\partial_{z_i} f + 2\Delta f = 0
\]
for $i = 1, 2$. The result follows. \qed

A similar argument works for three point functions.

### 3.4. More on the operator algebra.

#### 3.4.1. Secondary fields.

#### 3.4.2. Associativity.

#### 3.4.3. Conformal blocks.

### 4. Minimal Models

#### 4.1. Degenerate families.

#### 4.2. Fusion rules.
4.3. Ising model.

5. Modular Transformations and Verlinde Formula

5.1. CFT on a torus.

5.2. Characters.

5.3. Conformal blocks.

6. Bosonization

References


E-mail address: wentworth@jhu.edu