What’s the flux?

Once we have our definition of surface integrals

$$\int_{S} f d\sigma = \int_{\Omega} f(x(u,v), y(u,v), z(u,v)) |N(u,v)| \, dudv,$$

it can be applied to the case of a flux function, as in (17.7.9). By that, I just mean that the function $f$ to be integrated is given to be of the form $\mathbf{v} \cdot \mathbf{n}$, where $\mathbf{n}$ is a unit normal field to the surface. The point is to see that the amount of calculation in the case of a flux is actually less than in general.

*Unless $S$ is a closed surface, it is not possible to attach meaning to the outward normal, pointing in the direction away from the surface.* The same issue appears already for curves: if we are given the start of a simple closed curve

there is no way of knowing whether the point * will be inside or outside the curve until the rest of the curve is drawn.

We will have the same sort of simplification as we had with line integrals. Recall that if $\mathbf{r} = \mathbf{r}(t)$, with $t \in I$, gives a parametrization of the curve $C$, then

$$\int_{C} f ds = \int_{I} f(\mathbf{r}(t)) \| \mathbf{r}'(t) \| \, dt.$$  

So when $f ds = \mathbf{F} \cdot d\mathbf{r}$, we have

$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{I} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{\| \mathbf{r}'(t) \|} \| \mathbf{r}'(t) \| \, dt = \int_{I} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt.$$  

The vector version is simpler, with no annoying square roots due to taking the norm.

With that said, let $\mathbf{r} = \mathbf{r}(u,v)$, with $(u,v) \in \Omega$ be a parametrization of $S$. The point here is that we can write as usual:

$$\int_{S} (\mathbf{v} \cdot \mathbf{n}) d\sigma = \int_{\Omega} (\mathbf{v} \cdot \mathbf{n}) |N(u,v)| dudv.$$  

We established in lecture that $N$ is a normal field to the surface. Thus

$$\mathbf{n} = \pm \frac{\mathbf{N}}{\| \mathbf{N} \|}.$$  

Since $\mathbf{n}$ was given arbitrarily, we really have no way to decide the sign without additional information. Then, we can finally assert:

$$\int_{\Omega} (\mathbf{v} \cdot \mathbf{n}) |N(u,v)| dudv = \pm \int_{\Omega} (\mathbf{v} \cdot \mathbf{N}) dudv.$$  

In other words, the fancy vector integrals have *simpler* formulas.