Double trouble?

I was asked about 16.4 #11. I promised to post this, but I did have second thoughts (see #4 below). This is about determining the inequalities defining a region in polar coordinates. The way it is done is basically the same as determining the inequalities for rectangular coordinates. To be specific:

1. The region in question is \( \Omega = \{(x, y) : \frac{1}{2} \leq x \leq 1, 0 \leq y \leq \sqrt{1 - x^2}\} \).

2. If we take this at face value—do we ever do otherwise?—we see we are taking the intersection of the strip \( \frac{1}{2} \leq x \leq 1 \) with the upper half of the unit disc.

3. [Draw a picture!] The region \( \Omega \) can also be described as the one bounded by the three curves: the portion of the unit circle in the first quadrant, the line \( x = \frac{1}{2} \) and the line \( y = 0 \). The first two intersect in the point \( P = (1/2, \sqrt{3}/2) \).

4. We want to set up inequalities for the polar coordinates, familiar quantities from Calc II. At this point, the student who’s a little weak in polar coordinates should look at that material again; it’s in Ch. 9, and area in polar coordinates is specifically treated in 9.5. Shame on you if you needed to see this section again, but didn’t refer back to it! (See item #4 of the academic orientation document, To the Freshmen, which is part of the reading material of this course.)

5. It goes more or less as in #1 above. It is “usually” right to specify an interval for \( \theta \), and then an interval for \( r \) that depends on \( \theta \). The \( \theta \)-coordinate of the point \( P \) is \( \pi/3 \). Thus, in the region \( \Omega \), \( 0 \leq \theta \leq \pi/3 \). For each such \( \theta \), \( r \) goes “from the vertical line \( x = \frac{1}{2} \) to the unit circle,” which we understand to mean “... radially.”

6. The polar equation of the unit circle is \( r = 1 \). That’s easy. The polar equation of the line \( x = \frac{1}{2} \) is found by simply converting the latter to polar coordinates: \( r \cos \theta = \frac{1}{2} \), i.e.,

\[
r = \frac{1}{2 \cos \theta},
\]

which we may write if we wish: \( r = \frac{1}{2} \sec \theta \). In other words, in polar coordinates \( \Omega \) is given as:

\[\{(r, \theta) : 0 \leq \theta \leq \frac{\pi}{3}, \frac{1}{2} \sec \theta \leq r \leq 1\}\].

7. Isn’t this what we mean by changing variables (in this case, from \((x, y)\) to \((r, \theta)\))? For integration, we need only remember to replace “\(dydx\)” by “\(rdrd\theta\)”.