Section 9.5.

Question 2. Calculate the area $A$ enclosed by $r = a \cos 3\theta$ from $\theta = -\frac{1}{6} \pi$ to $\theta = \frac{1}{6} \pi$, where $a > 0$.

**Sol.**

$$A = \int_{-\frac{1}{6} \pi}^{\frac{1}{6} \pi} \frac{1}{2} a^2 \cos^2 3\theta d\theta = \int_{-\frac{1}{6} \pi}^{\frac{1}{6} \pi} \frac{1}{4} a^2 (1 + \cos 6\theta) d\theta = \frac{a^2}{4} \left[ (\theta + \frac{1}{6} \sin 6\theta) \right]_{-\frac{1}{6} \pi}^{\frac{1}{6} \pi} = \frac{a^2}{12} \pi.$$

Question 8. Calculate the area $A$ of the region defined by $r = \cos \theta$, $r = \sin \theta$, $\theta = 0$, and $\theta = \frac{\pi}{4}$.

**Sol.**

$$A = \int_{0}^{\frac{\pi}{4}} \frac{1}{2} \left( \cos^2 \theta - \sin^2 \theta \right) d\theta = \int_{0}^{\frac{\pi}{4}} \frac{1}{2} \cos 2\theta d\theta = \left[ \frac{1}{4} \sin 2\theta \right]_{0}^{\frac{\pi}{4}} = \frac{1}{4}.$$

Question 14. Find the area $A$ of the region defined by $r = e^\theta$, $2\pi \leq \theta \leq 3\pi$; $r = \theta$, $0 \leq \theta \leq \pi$; the rays $\theta = 0$, $\theta = \pi$.

**Sol.**

$$A = \frac{1}{2} \int_{2\pi}^{3\pi} e^{2\theta} d\theta - \frac{1}{2} \int_{0}^{\pi} \theta^2 d\theta = \frac{1}{2} \left[ \frac{1}{2} e^{2\theta} \right]_{2\pi}^{3\pi} - \frac{1}{2} \left[ \frac{1}{3} \theta^3 \right]_{0}^{\pi} = \frac{1}{4} (e^{6\theta} - e^{4\theta}) - \frac{1}{6} \pi^3.$$

Represent the indicated area $A$ by one or more integrals

Question 18. Outside $r = 1 - \cos \theta$, but inside $r = 1 + \cos \theta$.

**Sol.**

$$A = 2 \int_{0}^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos \theta)^2 - (1 - \cos \theta)^2 d\theta = \int_{0}^{\frac{\pi}{2}} 4 \cos \theta d\theta.$$

Question 24. Outside $r = 1 + \cos \theta$, but inside $r = 2 - \cos \theta$.

**Sol.**

$$A = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2 - \cos \theta)^2 - (1 + \cos \theta)^2 d\theta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 - 6 \cos \theta d\theta.$$

Question 26. Inside one petal of $r = 5 \cos 6\theta$.

**Sol.**

$$A = \frac{1}{2} \int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} (5 \cos 6\theta)^2 d\theta = \frac{25}{2} \int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} \cos^2 6\theta d\theta.$$
Section 9.6

Question 4. Express the curve \( x(t) = 2t - 1, \ y(t) = 8t^3 - 5 \) by an equation in \( x \) and \( y \).

**Sol.** \( t = \frac{x+1}{2} \). Therefore,

\[
y = 8 \left( \frac{x + 1}{2} \right)^3 - 5 = (x + 1)^3 - 5 = x^3 + 3x^2 + 3x - 4.
\]

\( \square \)

Question 10. Express the curve \( x(t) = e^t, \ y(t) = 4 - e^{2t} \) by an equation in \( x \) and \( y \).

**Sol.**

\[
y = 4 - (e^t)^2 = 4 - x^2.
\]

Since \( e^t > 0 \) for any \( t \), we have a restriction \( x > 0 \). \( \square \)

Express the curve by an equation in \( x \) and \( y \); then sketch the curve.

Question 14. \( x(t) = 3 \cos t, \ y(t) = 2 - \cos t, \ 0 \leq t \leq \pi \).

**Sol.** \( \cos t = \frac{1}{3} \), therefore,

\[
y = 2 - \cos t = 2 - \frac{1}{3}x.
\]

Since \(-3 \leq 3 \cos t \leq 3\) on \( 0 \leq t \leq \pi \), we have a restriction \(-3 \leq x \leq 3\).
Question 16. \( x(t) = \frac{1}{t}, \quad y(t) = \frac{1}{t^2}, \quad 0 < t < 3. \)

**Sol.**

\[
y = \left(\frac{1}{t}\right)^2 = x^2.
\]

Since \( \frac{1}{3} < \frac{1}{t} \) on \( 0 < t < 3 \), we have a restriction \( x > \frac{1}{3} \).

\[\begin{array}{c}
\text{y} \\
\hline
\text{x}
\end{array}\]

\((\frac{1}{3}, \frac{1}{9})\)

---

Question 20. \( x(t) = 2 \sin t, \quad y(t) = \cos t, \quad 0 \leq t \leq \frac{\pi}{2}. \)

**Sol.** \( \sin t = \frac{1}{2} x \) and \( \cos t = y \). Therefore,

\[
1 = \sin^2 t + \cos^2 t = \left(\frac{1}{2}x\right)^2 + y^2 = \frac{x^2}{4} + y^2.
\]

Because \( 0 \leq 2 \sin t \leq 2 \) on \( 0 \leq t \leq \frac{\pi}{2} \), we have a restriction \( 0 \leq x \leq 2 \).

\[\begin{array}{c}
\text{y} \\
\hline
\text{x}
\end{array}\]

\((0,1)\)

\((2,0)\)
Question 22. Parametrize: (a) the curve $y = f(x), x \in [a, b]$; (b) the polar curve $r = f(\theta), \theta \in [\alpha, \beta]$.

**Sol.** (a) $x(t) = t, y(t) = f(t), \text{ and } t \in [a, b]$.
(b) $x(t) = f(\theta) \cos \theta, y(t) = f(\theta) \sin \theta, \text{ and } \theta \in [\alpha, \beta]$. □

Question 24. A particle with position given by the equations

$$x(t) = 3 \cos 2\pi t, \quad y(t) = 4 \sin 2\pi t, \quad t \in [0, 1],$$

starts at the point $(3, 0)$ and traverses the ellipse $16x^2 + 9y^2 = 144$ once in a counterclockwise manner. Write equation of the form

$$x(t) = f(t), \quad y(t) = g(t), \quad t \in [0, 1],$$

so that the particle
(a) begins at $(3, 0)$ and traverses the ellipse once in a clockwise manner;
(b) begins at $(0, 4)$ and traverses the ellipse once in a clockwise manner;
(c) begins at $(-3, 0)$ and traverses the ellipse twice in a counterclockwise manner;
(d) traverses the upper half of the ellipse from $(3, 0)$ to $(-3, 0)$.

**Sol.** (a) $x(t) = -3 \cos 2\pi t, y(t) = 4 \sin 2\pi t$.
(b) $x(t) = 3 \sin 2\pi t, y(t) = 4 \cos 2\pi t$.
(c) $x(t) = -3 \cos 4\pi t, y(t) = -4 \sin 4\pi t$.
(d) $x(t) = 3 \cos \pi t, y(t) = 4 \sin \pi t$. □

Question 26. Find a parametrization

$$x(t) = \sin f(t), \quad y(t) = \cos f(t), \quad t \in (0, 1),$$

which traces out the unit circle infinitely often.

**Sol.** Since $\sin^2 f(t) + \cos^2 f(t) = 1$ for any $t \in (0, 1)$, it is enough to find a function $f(t)$ which maps $(0, 1)$ onto an infinite interval. For example, $f(t) = \frac{2\pi}{1-t}$ is such a function. □

Question 28. Find a parametrization $x = x(t), y = y(t), t \in [0, 1]$ for the line segment from $(2, 6)$ to $(6, 3)$.

**Sol.** $(x(t), y(t)) = (1-t)(2, 6) + t(6, 3) = (2-2t+6t, 6-6t+3t) = (2+4t, 6-3t)$. Therefore, $x(t) = 2+4t$, and $y(t) = 6 - 3t$. □

Question 32. Find a parametrization $x = x(t), y = y(t), t \in [0, 1]$ for the curve $y^3 = x^2$ from $(1, 1)$ to $(8, 4)$.
**Sol.** From $y^3 = x^2$, derive $y = x^{\frac{2}{3}}$. Then, we see that $x(t) = 7t + 1$, and $y(t) = (7t + 1)^{\frac{2}{3}}$. □