Cyclic

The notion of a $T$-cyclic subspace of $V$ is rather straightforward, but the consideration of such such subspaces is one important ingredient in understanding the “simplest form” of a linear transformation. Some other key ingredients in the process are: $T$-invariant spaces (in general), direct sums, and generalized eigenspaces. All but the last one have appeared already in the course.

Let $T : V \rightarrow V$ be a linear transformation of a vector space over some field $\mathbb{F}$, and let $v \in V$. The definition of the $T$-cyclic subspace $W$ of $V$ generated by $V$ is, in words, the smallest $T$-invariant subspace $V$ that contains $v$. What does that mean explicitly? $W$ must contain $v$, $Tv$, $T(Tv) = T^2v$, etc., and be closed under the two operations of linear algebra (vector addition and scalar multiplication). We can see that

$$\text{Span}\{T^\ell(v) \mid \ell \geq 0\}$$

is $T$-invariant, so that’s the $W$ we are taking about.

The main issue one has to address is: If one has a proper, non-trivial $T$-invariant subspace $W$, does it have a $T$-invariant complementary subspace? or better, how can we determine one (at least in principle)? The theorem (Jordan canonical form) says: Let $V$ be a finite-dimensional vector space, over a field in which every polynomial splits into a product of linear factors (e.g., $\mathbb{C}$). Let $T : V \rightarrow V$ be a linear transformation. Then $V$ can be decomposed into a direct sum of $T$-cyclic subspaces ($V = \bigoplus W_i$), on each of which the restriction $T_{W_i}$ of $T$ to $W_i$ has only one eigenvalue (with the $W_i$’s as small as possible). (Two such $W_i$’s could have the same single eigenvalue.)

Do the following problems:

1. Show that $W = P(T)v$, which means $\{w \in V \mid w = g(T)v \text{ for some } g \in \mathbb{F}(t)\}$.

2. Show that when $V$ is finite-dimensional, of dimension $n$,
   a) $W = \text{Span}\{T^\ell v \mid 0 \leq \ell < n\}$.
   b) Show that there is a unique $d < n$ for which $\beta = \{T^\ell v \mid \ell < d\}$ is a basis of $W$.
   c) For $d$ as in part b), show that there is a unique monic polynomial $g(t)$ (i.e., the leading coefficient of $g$ is equal to 1) of degree $d$ for which $g(T)v = 0$.
   d) Determine the matrix $[T]_\beta$ when $\beta$ from part b) is taken as basis of $W$.
   e) Let $A$ be the $2 \times 2$ matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Determine the $L_A$-cyclic subspaces of $\mathbb{F}^2$ generated by each of the following vectors:

   (i) $e_1$, (ii) $(e_1 + e_2)$, (iii) $e_2$, (iv) $(2e_1 + e_2)$.

   f) Determine all $L_A$-cyclic subspaces of $\mathbb{F}^2$ when $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Do this same for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. 

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