Excentric compactifications  Steven Zucker

The term *excentric* was coined by the author [6:§1],[13:§2]. It is accented on the first syllable, in contrast with the English word “eccentric”, and conveys the following idea. For now, let $W$ be a unipotent algebraic group. Then $W/W$ (the trivial group) is the *reductive* quotient of $W$. When $U \subseteq W$ is a subgroup that is the center of something, then $W/U$ is the (or an) excentric quotient of $W$.

We present the setting for these notes. Let $D$ be a symmetric space of non-compact type, and $\Gamma$ an arithmetically defined group of isometries of $D$; put informally, this means that some algebraic group $G$ over $\mathbb{Q}$ has its real points giving the isometry group of $D$, and $\Gamma$ is roughly $G(\mathbb{Z})$. If $\Gamma$ is not too big (i.e., is torsion-free, later neat), then $X = \Gamma \backslash D$ is a manifold. When $D$ has an invariant complex structure, $D$ is called *Hermitian*, as is $X$. The latter is called a locally symmetric variety, for $X$ is a quasi-projective complex algebraic variety [2].

Typically, $X$ is non-compact and one soon realizes that it is important to compactify it. There exist too many compactifications of $X$, so we select one or more to suit a given purpose. It is common enough to attach a $\Gamma$-equivariant boundary $\partial D$ to $D$, and then take the quotient by $\Gamma$. Here are two such compactifications of $X$:

i) $\overline{X} = \Gamma \backslash \overline{D}$, the manifold-with-corners of Borel-Serre [3],

ii) $X^{Sa} = \Gamma \backslash D^{Sa}$, a Satake compactification of $X$ [9] (see also [11]). There are finitely many Satake compactifications. When $X$ is Hermitian, one of these is topologically the Baily-Borel compactification $X^*$, a projective variety over $\mathbb{C}$ that is generally quite singular.

When $X$ is Hermitian, there are also the smooth toroidal compactifications $X^{tor}$ of Mumford et al. [1], constructed so that $\partial X^{tor}$ is a divisor with normal crossings. It is not unique in general, but rather depends on certain combinatorial data.

A morphism $Y_1 \rightarrow Y_2$ of compactifications of $X$ is the unique extension of the identity mapping of $X$, if it exists. For instance, for the three types of compactification above of a locally symmetric variety, there are morphisms

\[
X^{tor} \quad \downarrow
\]

\[
\overline{X} \rightarrow X^*.
\]

We see that $X^*$ is a common quotient of $\overline{X}$ and $X^{tor}$. In general, there is no morphism in either direction between $\overline{X}$ and $X^{tor}$.

One might take as a criterion for a good compactification that a (locally) homogeneous vector bundle $E \rightarrow X$ should extend to the compactification. Extending
to $\overline{X}$ is trivial, as $\overline{X}$ is homotopy equivalent to $X$. It is wiser to take a quotient $\overline{X}^\text{red}$ of $\overline{X}$, the reductive Borel-Serre compactification, which is defined as follows. The open faces of $\overline{D}$ are of the form

$$e(R) \simeq D_R \times W_R,$$

with $W_R$ the unipotent radical of $R$ (real points). To get the open faces of $\overline{D}^\text{red}$, one collapses $W_R$ to a point, yielding $e(R)^\text{red} \simeq D_R$. This is seen to define the reductive quotient $\overline{X}^\text{red}$ of $\overline{X}$, a stratified compactification of $X$. The bundle extension $\overline{E}^\text{red} \rightarrow \overline{X}^\text{red}$ can be carried out by performing the Borel-Serre construction on the total space of $E$ to produce $\overline{E} \rightarrow \overline{X}$, and then taking reductive quotients.

As for the extension of $E$ to $X^\text{tor}$, this was done by Mumford [8], but we can alternatively take here the toroidal construction on the total space of $E$.

How different are $\overline{X}^\text{red}$ and $X^\text{tor}$? There are two canonical notions (for compactifications of the same space): the greatest common quotient (GCQ) and the least common modification (LCM) [6]. These satisfy universal mapping properties:

$$
\begin{align*}
Y_1 \rightarrow & \quad \text{GCQ}(Y_1, Y_2) \leftarrow Y_2 \\
 & \quad Q \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quartes
\[ H^{\bullet}_{(\infty),gr}(X) \rightarrow H^{\bullet}_{(p),gr}(X) \simeq H^{\bullet}(X^{\text{tor}}) \quad (1 < p < \infty). \]

(The second line is different from the treatment in [8].) Furthermore, under the isomorphisms in the above, the Chern forms of an invariant connection map to the Chern classes of \( E^{\text{red}} \) and \( E^{\text{tor}} \) respectively.

Now is the time to bring in the excentric compactifications of \( X \). Let \( e(R) \) be, as before, the \( R \)-stratum of \( \overline{X} \) for the \( \mathbb{Q} \)-parabolic subgroup \( R \) of \( \mathcal{G} \), and let \( Z(R) \) denote the \( R \)-stratum of \( X^{\text{tor}} \). Both have an action of \( U_P \), the center of \( W_P \), when \( R \) is subordinate to \( P \); that means that \( P \) is the “smallest” maximal parabolic subgroup containing \( R \), and we have \( U_P \subseteq W_R \). In the toroidal case, the tori that occur are of the form \( T_P = \Gamma(U_P) \backslash U_P(\mathbb{C}) \). We take the quotients at the respective boundary strata,

\[ D_R \times W_R \simeq e(R) \rightarrow e(R)^{\text{exc}} =: e(R)/U_P \simeq D_R \times (W_R/U_P), \]

(recall the opening paragraph) and \( Z(R) \rightarrow Z(R)/U_P \), obtaining the excentric compactifications \( \overline{X}^{\text{exc}} \) (with morphisms \( \overline{X} \rightarrow \overline{X}^{\text{exc}} \rightarrow \overline{X}^{\text{red}} \)) and \( X^{\text{tor,exc}} \) (a quotient of \( X^{\text{tor}} \)). The two excentric quotients are still different in general, but less so than \( \overline{X}^{\text{red}} \) and \( X^{\text{tor}} \). For instance, one can see rather easily that the corresponding strata of \( \overline{X}^{\text{exc}} \) and \( X^{\text{tor,exc}} \) are homotopy equivalent.

There are bundle extensions \( \overline{E}^{\text{exc}} \rightarrow \overline{X}^{\text{exc}} \) (the pullback of \( \overline{E}^{\text{red}} \)) and \( E^{\text{tor,exc}} \rightarrow X^{\text{tor,exc}} \) (which pulls back to \( E^{\text{tor}} \)). We have the following analogue of Prop. 1 and Conj. 1:

**Proposition 2.** i) In the canonical diagram

\[
\begin{array}{ccc}
\text{LCM}(\overline{X}^{\text{exc}}, X^{\text{tor,exc}}) & \xrightarrow{\beta} & X^{\text{tor,exc}} \\
\downarrow{\alpha} & & \\
\overline{X}^{\text{exc}} & & \\
\end{array}
\]

both projections \( \alpha \) and \( \beta \) are homotopy equivalences.

ii) Let \( k : X^{\text{tor,exc}} \rightarrow \overline{X}^{\text{exc}} \) be the mapping defined by composing \( \alpha \) with a homotopy inverse to \( \beta \) in i). Then \( k^{*}E^{\text{exc}} \simeq E^{\text{tor,exc}} \).

**Corollary.** Conjecture 1 is true.

The corollary is an immediate consequence of (ii) in Prop. 2. We give some indication of the proof of Prop. 2 [13] in the following outline:

1. The proof of the assertion in (i) about \( \beta \) goes, more or less, like the argument in [4] (for (ii) in Prop. 1 above). We show that \( \beta \) has contractible fibers.
2. From (*), we get

\[ X^\text{tor,exc} \rightarrow X^\text{exc} \rightarrow X^*. \]

The problem of determining the fibers of \( \beta \) fibers over \( X^* \). This brings in partial compactifications of homogeneous cones, and then the duality noted in [5.§2.3].

3. The means for deducing the assertion in (i) about \( \alpha \) goes under the name LCM-basechange. This is a rather simple notion. Suppose that \( Y_1 \rightarrow Y_2 \) is a morphism of compactifications of a space \( X \), and that \( Y_3 \) is a third compactification of \( X \). It is easy to see that one has an inclusion

\[ \text{LCM}(Y_1, Y_3) \subseteq Y_1 \times_{Y_2} \text{LCM}(Y_2, Y_3). \]

We say that LCM-basechange holds in the given situation if the inclusion is an equality. In that case, the projections \( \text{LCM}(Y_1, Y_3) \rightarrow Y_1 \) and \( \text{LCM}(Y_2, Y_3) \rightarrow Y_2 \) have the same fiber.

4. Statement (ii) is verified directly.

References


