For the first 12 problems we will be working with linear transformations $L : \mathbb{R}^4 \to \mathbb{R}^4$ and $T : \mathbb{R}^4 \to \mathbb{R}^4$. We have the standard basis $e_i$ and another basis $v_i$. We define $L$ by $L(e_i) = v_i$ and $T$ by $T(v_1) = v_1 + v_2$, $T(v_2) = v_2 + v_3$, $T(v_3) = v_3 + v_4$, and $T(v_4) = v_4 + v_1$. We also are given that the determinant for the matrix $(v_1, v_2, v_3, v_4)$ is 2.

(1) (3 points) What is the matrix for $L$ with respect to (w.r.t.) the standard basis $e_i$?

(2) (3 points) What is the matrix for $T \circ L$ w.r.t. the standard basis $e_i$?

(3) (3 points) What is the matrix for $T$ w.r.t. the basis $v_i$?
(4) (3 points) What is the determinant of $L$?

(5) (3 points) What is the determinant of $T$?

(6) (3 points) What is the determinant of $T \circ L$?
(7) (3 points) Find a basis for the image of $L$.

(8) (3 points) Find a basis for the image of $T$.

(9) (3 points) Find a basis for the image of $T \circ L$. 
(10) (3 points) Find a basis for the kernel of $L$.

(11) (3 points) Find a basis for the kernel of $T$.

(12) (3 points) Find a basis for the kernel of $T \circ L$. 
For the next 7 problems we will consider the linear transformation $D : P_2 \rightarrow P_2$ given by $D(f) = f'$, the derivative. We will consider $P_1 \subset P_2 \subset C[-1,1]$ with the usual inner product.

(13) (3 points) What is the matrix for this linear transformation with respect to the basis $\{1, x, x^2\}$?

(14) (3 points) Find an orthonormal basis for $P_1$ starting with $\frac{1}{\sqrt{2}}$. 
(15) (3 points) Find a non-zero polynomial in \( P_2 \) perpendicular to \( P_1 \).
(16) (3 points) Find an orthonormal basis for $P_2$. (Hint: The first and third vectors are $\frac{1}{\sqrt{2}}$ and $\frac{3\sqrt{2}}{2\sqrt{2}}(x^2-\frac{1}{3})$. However, your work has to derive these and you will be graded on your work, not on your answer (although it must be correct). In particular, you cannot start with these and prove they are the answer, you must derive them.)
(17) (3 points) What is the matrix, $A$, for our linear transformation $D : P_2 \rightarrow P_2$ with respect to our orthonormal basis from problem (16)?
(18) (3 points) Find orthonormal $v_i$ and $u_i$ and singular values $\sigma_i$ so that $D(v_i) = \sigma_i u_i$ and $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$. 
(19) (3 points) This could be quite difficult but if you know what you are doing you can read this off from the previous problem. Find the singular value decomposition for the matrix $A$ of problem (17). I.e., write $A = U \Sigma V^T$ where $U$ and $V$ are orthonormal and $\Sigma$ is diagonal as in the theorem.
(20) (3 points) Find the Eigenvalues for
\[
\begin{pmatrix}
\frac{5}{2} & 0 & \frac{3}{2} \\
0 & 1 & 0 \\
\frac{3}{2} & 0 & \frac{5}{2}
\end{pmatrix}.
\]
(21) (3 points) Find an orthonormal Eigenbasis for the matrix \[
\begin{pmatrix}
\frac{5}{2} & 0 & \frac{3}{2} \\
0 & 1 & 0 \\
\frac{3}{2} & 0 & \frac{5}{2}
\end{pmatrix}
\]. Order your vectors by taking the highest Eigenvalue first.
(22) (3 points) Find the singular values for the matrix, $A = \begin{pmatrix} \sqrt{2} & 0 & \sqrt{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix}$. 
(23) (3 points) Find orthonormal $v_i$ and $u_i$ and singular values $\sigma_i$ so that $A(v_i) = \sigma_i u_i$ and $\sigma_1 \geq \sigma_2 \geq \sigma_3$ for the $A$ of problem (22).
(24) (3 points) Find the singular value decomposition for the matrix $A$ of problem (22). I.e., write $A = U \Sigma V^T$ where $U$ and $V$ are orthonormal and $\Sigma$ is diagonal as in the theorem.
(25) (3 points) Consider the quadratic form \( q(x_1, x_2, x_3) = \frac{5}{2}x_1^2 + x_2^2 + \frac{5}{2}x_3^2 + 3x_1x_3 \). Find the matrix for this quadratic form.

(26) (3 points) You can change the basis for the quadratic form in problem (25) to a new orthonormal basis where the quadratic form has no cross terms. Using these new coordinates, what is \( q(c_1, c_2, c_3) \)?
(27) (3 points) If we set \( q(x_1, x_2, x_3) = 1 \) for the quadratic form of problem (25) we get a surface in \( \mathbb{R}^3 \). How far from the origin is the closest point in this set?

(28) (3 points) Using standard coordinates, what two points are the closest to the origin?
(29) (3 points) The equation \( q(x_1, x_2, x_3) = 1 \) for the quadratic form of problem (25) represents a surface in \( \mathbb{R}^3 \). In problem (28) you found the two points closest to the origin. There is a whole circle of points on this surface that are furthermost from the origin. What is the radius of this circle?

(30) (3 points) Give an equation for the entire circle of points furthermost from the origin in the surface given by the equation \( q(x_1, x_2, x_3) = 1 \) for the quadratic form of problem (25). Use the standard coordinates.
(31) (6 points) Find the degree 2 (or less) polynomial $y = f(x) = a + bx + cx^2$ which is the best fit for the scatter plot given by the pairs of points $(x, y) = (-1, 0), (0, 1), (1, -1), (2, 0)$. 
More paper for problem (31).