1. [25pts] Let $f(x, y) = \ln(x - y^2)$.

(a) Find the domain and range of $f(x, y)$.

Since the domain of $\ln(t)$ is all values of $t > 0$, we have:

$$Domain = \{(x, y) \in \mathbb{R}^2 : x - y^2 > 0\}$$
$$= \{(x, y) \in \mathbb{R}^2 : x > y^2\}$$

While it wasn’t necessary to sketch the domain in the $xy$-plane, here is a diagram showing the region. Note, the domain is shaded in blue, with an open (dashed) boundary, since $\ln$ is undefined for zero.

For the range, note that when $y = 0$, we have $f(x, 0) = \ln(x)$, and since $(x, 0)$ is in our domain for all positive values of $x$, we must have that the
range of $f(x, y)$ is the same as the range of $\ln(x)$. That is:

$$\text{Range} = \{ c \in \mathbb{R} : -\infty < c < \infty \},$$
i.e. the interval $(-\infty, \infty)$.

(b) Determine the equations of the level curves $f(x, y) = c$ together with possible values of $c$.

The possible values of $c$ are exactly the values in the range. Hence $-\infty < c < \infty$. Setting $f(x, y) = c$ gives us:

$$\ln(x - y^2) = c \Rightarrow x - y^2 = e^c \Rightarrow x = y^2 + e^c$$

Since $0 < e^c < \infty$ for $-\infty < c < \infty$, then the level curves are exactly copies of the (sideways) parabola $x = y^2$, shifted to the right by the positive value $e^c$.

(c) Find a unit vector $\vec{u}$ which is perpendicular to the level curve of $f(x, y)$ passing through the point $(4, 1)$.

The gradient vector $\nabla f(x_0, y_0)$ will always be perpendicular to the level curve that passes through the point $(x_0, y_0)$:

$$\nabla f(4, 1) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)_{(4,1)} = \left( \frac{1}{x - y^2}, \frac{-2y}{x - y^2} \right)_{(4,1)} = (1/3, -2/3).$$

Hence a unit vector in the same direction as $\nabla f(x, y)$ will also be perpendicular to the level curve through $(4, 1)$:

$$\vec{u} = \frac{\nabla f(4, 1)}{||\nabla f(4, 1)||} = \frac{3}{\sqrt{5}} (1/3, -2/3) = \frac{1}{\sqrt{5}} (1, -2)$$
2. [20 points] Let \( f(x, y) = e^{x^2} \sin(xy) \) be a function of two variables.

(a) Find the linearization of \( f(x, y) \) at the point \((1, 0)\).

\[
f(1, 0) = 0
\]

\[
\frac{\partial f}{\partial x}\bigg|_{(1,0)} = 2xe^{x^2} \sin(xy) + e^{x^2} y \cos(xy)\bigg|_{(1,0)}
\]

\[
= e^{x^2}(2x \sin(xy) + y \cos(xy))\bigg|_{(1,0)}
\]

\[
= 0
\]

\[
\frac{\partial f}{\partial y}\bigg|_{(1,0)} = xe^{x^2} \cos(xy)\bigg|_{(1,0)}
\]

\[
= e
\]

Hence the linearization of \( f(x, y) \) at \((1, 0)\) is:

\[
L(x, y) = f(1, 0) + \frac{\partial f}{\partial x}\bigg|_{(1,0)} (x - 1) + \frac{\partial f}{\partial y}\bigg|_{(1,0)} (y - 0)
\]

\[
= 0 + (0)(x - 1) + (e)(y - 0)
\]

\[
= ey
\]

(b) Use your answer in part (a) to approximate \( f(1.1, -0.05) \).

\[
f(1.1, -0.05) \approx L(1.1, -0.05)
\]

\[
= -0.05e,
\]

which is about \(2.718 \times -0.05 = -0.1359\). (The actual value is \(-0.18434868...\)).
3. [15 points] Let \( w = \sin(f(x, y)) \) be a function of \( x, y \) and \( x = u(t), \ y = v(t) \) be functions of \( t \). Find \( \frac{dw}{dt} \) in terms of \( f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, u, v, u', v' \).

\[
\frac{dw}{dt} = \frac{d}{dt} \sin(f(u(t), v(t))) \\
= \cos(f(u(t), v(t))) \frac{d}{dt} f(u(t), v(t)) \\
= \cos(f(u(t), v(t))) \left( \frac{\partial f}{\partial x}(u(t), v(t)) \frac{dx}{dt} + \frac{\partial f}{\partial y}(u(t), v(t)) \frac{dy}{dt} \right) \\
= \left( \cos(f(u(t), v(t))) \right) \left( \frac{\partial f}{\partial x}(u(t), v(t)) u'(t) + \frac{\partial f}{\partial y}(u(t), v(t)) v'(t) \right)
\]
4. [20 points] Find absolute minimum and maximum of the function \( f(x, y) = 12xy - 4x^2y - 3xy^2 \) on the triangle bounded by the \( x \)-axis, \( y \)-axis and the line \( 4x + 3y = 12 \).

i. 

\[ \nabla f(x, y) = (12y - 8xy - 3y^2, 12x - 4x^2 - 6xy) \]
\[ = (y(12 - 8x - 3y), x(12 - 4x - 6y)) \]

Setting \( \nabla f(x, y) = (0, 0) \) gives us

\[ y(12 - 8x - 3y) = 0 \]
\[ x(12 - 4x - 6y) = 0. \]

The first equation gives us \( y = 0 \) or \( 12 - 8x - 3y = 0 \). Letting \( y = 0 \), and substituting into the second equation gives us

\[ x(12 - 4x - 3(0)) = 0 \]
\[ \Rightarrow 4x(3 - x) = 0 \]
\[ \Rightarrow x = 0 \text{ or } x = 3. \]

Hence \((0, 0)\) and \((3, 0)\) are two critical points (and candidates for minima/maxima).

Letting \( 12 - 8x - 3y = 0 \) gives us \( y = 4 - \frac{8}{3}x \). Substituting into the second equation gives us

\[ x(12 - 4x - 6(4 - \frac{8}{3}x)) = 0 \]
\[ \Rightarrow 12x(x - 1) = 0 \]
\[ \Rightarrow x = 0 \text{ or } x = 1 \]
\[ \Rightarrow y = 4 \text{ or } y = \frac{4}{3}. \]

Hence \((0, 4)\) and \((1, \frac{4}{3})\) are two critical points (and candidates for maxima/minima).

ii. Along the \( x \)-axis we have \( f(x, y) = f(x, 0) \), with

\[ f(x, 0) = 12x(0) - 4x^2(0) - 3x(0)^2 \]
\[ = 0. \]
Hence $\frac{d}{dx} f(x, 0) = 0$ everywhere along the $x$-axis, and so all points $(x, 0)$ are critical points along the boundary (and candidates for maxima/minima).

Along the $y$-axis we have $f(x, y) = f(0, y)$, with

$$f(0, y) = 12(0)y - 4(0)^2y - 3(0)y^2 = 0.$$  

Hence $\frac{d}{dy} f(0, y) = 0$ everywhere along the $y$-axis, and so all points $(0, y)$ are critical points along the boundary (and candidates for maxima/minima).

Along the line $4x + 3y = 12$ we have $y = 4 - \frac{4}{3}x$, and $f(x, y) = f(x, 4 - \frac{4}{3}x)$, with

$$f(x, 4 - \frac{4}{3}x) = 12x(4 - \frac{4}{3}x) - 4x^2(4 - \frac{4}{3}x) - 3x(4 - \frac{4}{3}x)^2 = 0.$$  

Hence $\frac{d}{dx} f(x, 4 - \frac{4}{3}x) = 0$ everywhere along the line $4x + 3y = 12$, and so all points $(x, 4 - \frac{4}{3}x)$ are critical points along the boundary (and candidates for maxima/minima).

iii. Finally, we must test all the critical points both in the domain and along the boundary:

$$f(0, 0) = 0$$
$$f(3, 0) = 0$$
$$f(0, 4) = 0$$
$$f(1, \frac{4}{3}) = \frac{16}{3}$$
$$f(x, 0) = 0$$
$$f(0, y) = 0$$
$$f(x, 4 - \frac{4}{3}x) = 0$$

Hence there is an absolute maximum of $\frac{16}{3}$ at the point $(1, \frac{4}{3})$, and an absolute minimum of 0 along the entire boundary.
5. [20 points] Given the system

\[
\begin{align*}
\frac{dx}{dt} &= 2x + y \\
\frac{dy}{dt} &= 2y
\end{align*}
\]

(a) Write the system in the matrix form \( \frac{d\vec{x}}{dt} = A\vec{x}(t) \) Find all eigenvalues and corresponding eigenvectors of \( A \).

\[
\begin{pmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{pmatrix} = 
\begin{bmatrix}
2 & 1 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

\[
\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} = (2 - \lambda)^2 = 0
\]

Hence \( \lambda = 2 \) is the only eigenvalue of \( A \). To find the corresponding eigenvector,

\[
\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} = 2
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]

\[
\implies 2u_1 + u_2 = 2u_1 \implies u_2 = 0.
\]

Hence \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) is an eigenvector corresponding to the eigenvalue 2.

(b) Show that \( \vec{x}(t) = e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + te^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) is solution of the system.

\[
e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + te^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} te^{2t} \\ e^{2t} \end{bmatrix}
\]
Hence

\[
\frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} e^{2t} + t(2e^{2t}) \\ 2e^{2t} \end{bmatrix}
\]

\[
= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} te^{2t} \\ e^{2t} \end{bmatrix}
\]

\[
= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.
\]

Hence this is indeed a solution of the (matrix) differential equation.