Research Statement

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My main research interest is twofold. First I am interested in Harmonic Analysis on manifolds. More precisely, in my thesis, I studied the $L^\infty$ estimates and gradient estimates for the eigenfunctions of the Dirichlet and Neumann Laplacian on compact manifolds with boundary, and using these estimates, obtained the Hörmander multiplier theorem in this setting. Moreover, I am also interested in degenerate Fourier integral operators, Gibbs’ phenomenon, and Pinsky’s phenomenon for Fourier inversions and eigenfunction expansions. My second area of interest is Nonlinear Differential Equations, such as, well-posedness problems for nonlinear hyperbolic differential equations on manifolds, fully nonlinear equations, periodic solutions, subharmonics and homoclinic orbits for Hamiltonian systems. See: http://www.math.jhu.edu/~xxu/papers_and_preprints.html

1. My thesis work

In my thesis, for a compact Riemannian manifold $(M, g)$ with boundary, consider the Dirichlet (or Neumann) eigenvalue problem:

$$\begin{align*}
(\Delta + \lambda^2)u(x) &= 0, \quad x \in M, \\
u(x) &= 0, \quad x \in \partial M, \\
(\text{or } \partial_n u(x) &= 0 \quad x \in \partial M. \quad (1)
\end{align*}$$

Let $0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \lambda_3^2 \leq \cdots$ denote the eigenvalues of (1) and let $\{e_j(x)\} \subset L^2(M)$ be an associated real orthogonal basis with unit $L^2$ norm. Define the unit band spectral projection operators,

$$\chi_{\lambda}f = \sum_{\lambda_\ell \in [\lambda] \lambda + 1)} e_j(f), \quad \text{where} \quad e_j(f)(x) = e_j(x)\int_M f(y)e_j(y)dy.$$

The study of $L^p$ estimates for the eigenfunctions on compact manifolds has a long history. In the case of manifolds without boundary, Sogge [S0] showed the most general results of the form

$$||\chi_{\lambda}f||_p \leq C\lambda^{\sigma(p)}||f||_2, \quad \lambda \geq 1, \quad p \geq 2, \quad (2)$$

where

$$\sigma(p) = \max\left\{\frac{n-1}{2} - \frac{n}{p}, \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p})\right\}.$$

These estimates can’t be improved since $\limsup_{\lambda \to \infty} \lambda^{-\sigma(p)}||\chi_{\lambda}||_{L^2 \to L^p} > 0$ (see [S2]).
The special case of (2) where $p = \infty$ can be proved using the estimates of Hörmander [H1] that proved the sharp Weyl formula for general self-adjoint elliptic operators on manifolds without boundary. In case of manifolds with boundary, Grieser [G1] for two dimensional case, and Smith and Sogge [SS1] for the higher dimensional case, showed that the bounds (2) hold under the assumption that the manifold has geodesically concave boundary. Recently, without this assumption, Grieser [G] proved

$$||e_j(f)||_{\infty} \leq C \lambda^{(n-1)/2} ||f||_2, \quad \lambda \geq 1,$$

and Sogge [S3] proved the following $L^\infty$ estimate for $\chi_\lambda f$

$$||\chi_\lambda f||_{\infty} \leq C \lambda^{(n-1)/2} ||f||_2, \quad \lambda \geq 1,$$

which are sharp (for instance, when $M$ is the upper hemisphere of $S^n$ with the standard metric). One of my main results is the gradient estimates on the unit band spectral projection operators $\chi_\lambda f$:

**Theorem 1.** ([X1]) Fix a compact Riemannian manifold $(M, g)$ of dimension $n \geq 2$ with boundary, then there is a uniform constant $C$ so that

$$||\nabla \chi_\lambda f||_{\infty} \leq C \lambda^{(n+1)/2} ||f||_2, \quad \lambda \geq 1,$$

which is equivalent to

$$\sum_{\lambda_j \in [\lambda, \lambda + 1)} ||\nabla e_j(x)||^2 \leq C \lambda^{n+1}, \quad \forall x \in M.$$

The other main result will be on the Hörmander multiplier theorem for smooth compact Riemannian manifolds with boundary. Given a bounded function $m(\lambda) \in L^\infty(\mathbb{R})$, we can define the multiplier operator $m(P)$, by

$$m(P)f = \sum_{j=1}^{\infty} m(\lambda_j) e_j(f). \quad (3)$$

The Hörmander multiplier theorem addresses the problem of what smoothness assumption on the function $m(\lambda)$ is needed to ensure the boundedness of multiplier operator

$$m(P) : L^p(M) \rightarrow L^p(M), \quad 1 < p < \infty. \quad (4)$$

Many authors have studied the Hörmander multiplier theorem under different settings. Specifically, one assumes the following regularity assumption: Suppose that $m \in L^\infty(\mathbb{R})$. Let $L^2_s(\mathbb{R})$ denote the usual Sobolev space, and fix $\beta \in C_0^\infty((\frac{1}{2}, 2))$ satisfying $\sum_{-\infty}^{\infty} \beta(2^t) = 1$ for $t > 0$. Suppose also that for some real number $s > n/2$,

$$\sup_{\lambda > 0} \lambda^{-1+s} ||\beta(\cdot/\lambda)m(\cdot)||^2_{L^2_s} = \sup_{\lambda > 0} ||\beta(\cdot)m(\lambda \cdot)||^2_{L^2_s} < \infty. \quad (5)$$

Hörmander [H2] first proved the Hörmander multiplier theorem for $\mathbb{R}^n$ under the assumption (5). Stein [ST] and Stein and Weiss [SW] studied the Hörmander multiplier theorem for multiple Fourier series, which can be regarded as the case
for a flat torus $T^n$. Seeger and Sogge [SS] and Sogge [S2] proved the Hörmander multiplier theorem for compact manifolds without boundary under the assumption (5). Using the $L^\infty$ estimates on $\chi_\lambda f$ and $\nabla \chi_\lambda f$ and the ideas in [SS], [S1]-[S3], we have the following Hörmander multiplier theorem for compact manifolds with boundary:

**Theorem 2.** ([X2]) Let $m \in L^\infty(\mathbb{R})$ satisfy (5). Then there are constants $C_p$ such that

$$|||m(\cal{P})f|||_{L^p(M)} \leq C_p|||f|||_{L^p(M)}, \quad 1 < p < \infty. \quad (6)$$

Another result is for an important class of multiplier operator of spectral expansions, the Riesz means:

$$S^s_\delta f(x) = \sum_{\lambda_j \leq \lambda} \left(1 - \frac{\lambda_j^2}{\lambda^2}\right)^\delta e_j(f)(x). \quad (7)$$

Stein and Weiss [SW] studied the Riesz means for multiple Fourier series, which can be regarded as the case for a flat torus $T^n$. Sogge [S1] and Christ and Sogge [CS] proved the sharp results for compact manifolds without boundary, where the Riesz means (7) are uniformly bounded on all $L^p(M)$ provided that $\delta > \frac{n-1}{2}$, but no such result can hold for all $L^p(M)$ when $\delta \leq \frac{n-1}{2}$. Recently, Sogge [S3] proved the same results for the Riesz means of the spectral expansion for Dirichlet Laplacian on manifolds with boundary. Here is our result on the pointwise convergence for the Riesz means of the spectral expansions:

**Theorem 3.** ([X3]) Fix a smooth compact Riemannian manifold $(M, g)$ with boundary, let $f \in L^p(M)$, $1 \leq p \leq \infty$, for any $s > n/2$, we have

$$\lim_{\lambda \to \infty} S^s_\lambda f(x) = f(x), \quad \text{almost everywhere.}$$

2. Results on Hamiltonian systems

My earliest research experiences go back to my work on Hamiltonian systems with Professor Yiming Long, my Master thesis advisor; I continue to be interested in them. Let $H(t, u) \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ and consider the Hamiltonian system

$$(HS) \quad J \dot{u} + \nabla H(t, u) = 0, \quad (t, u) \in S_T \times \mathbb{R}^{2N},$$

where $J = \left( \begin{array}{cc} 0 & -I_N \\ I_N & 0 \end{array} \right)$ denotes the standard symplectic matrix, and $H(t, u)$ is $T$-periodic in $t$ and satisfies the following super-quadratic condition in the $u$-variable:

$$(H1) \text{ there is a constant } \mu > 2 \text{ such that}$$

$$0 < \mu H(t, u) \leq z \nabla H(t, u) \quad \text{for } |u| \text{ large.}$$
In his pioneer work [R1], Rabinowitz first proved the existence of periodic solutions for autonomous Hamiltonian systems under (H1). Afterwards, there were many studies on the existence of periodic solutions, subharmonics and homoclinic orbits for the nonautonomous Hamiltonian systems (HS) under some further conditions on the potential \( H \), (cf. [Ek], [Lo] and [R2]). In most papers on these problems, they required:

\[ (H)_p \text{ There are constants } c > 0 \text{ and } p \in (1, 2), \text{ such that } \]
\[ |\nabla H(t, u)|^p \leq c(\nabla H(t, u), u), \text{ for } |u| \text{ large}, \]

which implies that the potential \( H \) grows like \(|u|^q\) for some \( q \geq \mu \). This assumption on the potential \( H \) is required to show the compactness condition when one applies the critical point theory. Since one has \( H(u(t)) = \text{constant} \) for a solution \( u(t) \) of autonomous Hamiltonian systems and \( u(t) \) has the \( C^1 \) bound for the potential \( H \) satisfying (H1), one can expect that similar results should hold for the nonautonomous Hamiltonian systems. In [LX], [MTH]-[Xu7], two a priori estimates on \( C^1 \) bounds on the solutions of (HS) were proved for the much larger class potentials \( H \) satisfying:

\[ (H2) \text{ There is a constant } c > 0, \text{ such that } \]
\[ |\nabla H(t, u)| \leq c(\nabla H(t, u), u), \text{ for } |u| \text{ large}. \]

or

\[ (H3) \limsup_{|u| \to \infty} \frac{H_t(t, u)}{|u|^p H(t, u)} = 0, \text{ or } \liminf_{|u| \to \infty} \frac{H_t(t, u)}{|u|^p H(t, u)} = 0, \text{ uniformly in } t. \]

which includes these potentials grows like \( e^{|u|^k} \), where the condition (H2) only requires the angle between the directions \( \frac{\nabla H}{\|\nabla H\|} \) and \( \frac{u}{|u|} \) is \( O(1/|u|) \); and the condition (H3) measures how far the system (HS) is away from the autonomous system. The system (HS) with condition (H3) may be considered as a perturbed system with a large perturbation.

Applying these a priori estimates on solutions, in [LX], [MTH] and [Xu3], I demonstrated the existence of and multiplicity of periodic solutions of the system (HS) for a potential \( H \) with symmetry and without symmetry. In [Xu2] and [Xu7] I proved the existence and multiplicity of the homoclinic orbits of the system (HS) for a potential with symmetry and without symmetry, and for the convex potential. In [Xu1], [Xu4] and [Xu6] I demonstrated the existence of subharmonics for the system (HS), proved uniform \( C^1 \) estimates on subharmonics for the potentials \( H \) with the global super-quadratic condition, and studied the asymptotic behaviors of these subharmonics.

3. Future Research Plans

My research has dealt with and will continue to deal with areas such as harmonic analysis on manifolds, the inverse spectral problem, nonlinear differential equations
and Hamiltonian systems. I intend to continue to pursue several different vital problems in these areas.

1. Generalize the harmonic analysis on Euclidean spaces (cf. [ST1], [SW]) to general Riemannian manifolds. More precisely, study the $L^2$ restriction theorem, Riesz means, and general multiplier problems on Riemannian manifolds. In [S2], for manifolds without boundary, these problems were solved by studying the wave kernel using the parametrix construction. In my thesis, I studied the multiplier problem for manifolds with boundary. I plan to study some problems relative to the $L^2$ restriction theorem for manifolds with boundary and Bochner-Riesz type multiplier for cones.

2. Study the estimates of Laplacian eigenfunctions and spectrum for domains in $\mathbb{R}^n$ or compact manifolds with respect to some global geometry property, such as curvature and geodesic flow. These relate to the inverse spectral problems (cf. [M2], [SZ], [TZ], [Z1-3]). I hope to determine whether the results in [SZ] and [TZ] are true for manifolds with boundary. Moreover, I’d like to study the wave kernel for manifolds with boundary and with singularity.

3. The eigenfunction estimates and multiplier problems for domains in $\mathbb{R}^n$ or compact manifolds with rough boundary, and those elliptic operators with irregular coefficients. What I study in my thesis is the $C^\infty$ category. In [BI] and [IV], the sharp spectral asymptotic for operators with irregular coefficients was studied. It seems one can use their ideas to study local Weyl estimates in this setting, the key step in getting eigenfunction estimates.

4. Bilinear and multilinear eigenfunction estimates for Laplacian spectral projectors on manifolds with boundary, and their applications to nonlinear Schrödinger equations. In [BGT1-2], these estimates were proved for manifolds without boundary and were applied to nonlinear Schrödinger equations. I hope one may get similar results for manifolds with boundary.

5. Study degenerate Fourier integral operators (FIOs), Gibbs’ phenomenon, Pinsky’s phenomenon and decay rates for Fourier inversion on domains in $\mathbb{R}^n$ and spectral expansions on manifolds. For degenerate FIOs, there are many studies already (cf. [GS], [PS1-2], [ST2]). I want to find the decay estimates for FIOs with $C^\infty$ degenerate phase functions and study the generalized Radon transforms and Hilbert transforms. In [P], [PT] and [T], Gibbs’ phenomenon and Pinsky’s phenomenon were studied for Fourier inversion and spectral expansions on some special manifolds. I plan to study Gibbs’ phenomenon and Pinsky’s phenomenon for general Riemannian manifolds.

6. Study the existence, multiplicity and stability of solutions for Hamiltonian systems. It is still an open question as to the existence and multiplicity of periodic solutions for general first order and second order nonautonomous Hamiltonian systems (cf. [Ek], [Lo], [R3]). There are many interesting problems related to Hamiltonian systems such as the Rabinowitz Conjecture for the minimal period of autonomous systems, periodic Lagrangian orbits on tori, more generally, the periodic Lagrangian orbits on Riemannian manifolds and the periodic orbits of Hamiltonian system on the cotangent space $T^*M$ of some Riemannian manifolds $(M, g)$. 
References for Harmonic Analysis on Manifolds


[X1] Xiangjin Xu, Gradient estimates for eigenfunctions of Riemannian manifolds with boundary. (preprint)

[X2] Xiangjin Xu, Gradient estimates for eigenfunctions of compact manifolds with boundary and the Hörmander multiplier theorem. (preprint)

[X3] Xiangjin Xu, Spectral Expansions of Piecewise Smooth Functions on compact Riemannian manifolds with boundary. (preprint)


References for Hamiltonian systems:


[Xu7] Xiangjin Xu, Homoclinic orbits for first order Hamiltonian systems with convex potentials. (preprint)