Problem Set 2

Due at the beginning of class, February 5, 2007.

Relevant section of the text: Chapters 2 and 3.

1. Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}$ be a linear transformation such that

$$
T((1,0,0,0)) = 1, \quad T((1,-1,0,0)) = 0, \quad T((1,-1,1,0)) = 1, \quad T((1,-1,1,-1)) = 0.
$$

Determine $T((a,b,c,d))$. Find a basis for the kernel of $T$.

2. Let $V$ be a vector space that is countably generated, i.e., $V$ has a countable set of generators. (Recall that a countable set is one which is either finite or which can be put into one-to-one correspondence with the natural numbers.) Without the use of Zorn’s Lemma (or any of its equivalent forms!), prove that every linearly independent subset of $V$ can be extended to a (countable) basis of $V$.

3. Exhibit a finite-dimensional vector space $V$ of dimension 5, a subspace $W \subset V$ of dimension 3, and a basis of $V$ that contains no basis of $W$. [Question: Is there anything special about our choice of dimensions?]

4. Exactly how many 1-dimensional subspaces are there in a 3-dimensional vector space over the finite field of 5 elements? Prove that your answer is correct. [Can you generalize your answer to the setting of an arbitrary finite-dimensional vector space over an arbitrary finite field?]

5. Let $S$ and $T$ be subsets of a vector space $V$. For any subset $R \subset V$, denote by $\text{Span}(R)$ the linear span of $R$ (this is a subspace of $V$). Prove that $\text{Span}(S \cap T)$ is a subspace of $\text{Span}(S) \cap \text{Span}(T)$. By way of example, show that the inclusion may be proper.

6. Prove that all the vector spaces $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, \ldots are isomorphic when regarded as vector spaces over the rational numbers. [One strategy would be to show that they are all isomorphic to $\mathbb{R}$ as a vector space over $\mathbb{Q}$.]

7. Let $T$ be an idempotent linear operator of a finite-dimensional vector space $V$. (“Idempotent” means that $T^2 = T$.) Let $W$ and $N$ be the image and kernel of $T$, respectively. Prove the following assertions.

(a) $W \cap N = 0$.

(b) $V = W \oplus N$.

(c) There is a basis of $V$ with respect to which the matrix of $T$ has zeros and ones on the diagonal and zeros elsewhere.

Conversely, show that for an arbitrary subspace $W \subset V$, there is an idempotent linear operator $T: V \rightarrow V$ whose image is $W$. Is there a unique such operator? Justify your answer.
8. Let $V = \mathbb{R}[X]$ be the (real) vector space of polynomials with real-number coefficients. Let $D$ be the operator of differentiation, and let $T = p(X) \mapsto Xp'(X)$ (where $p'(X)$ is the derivative of $p(X)$).

(a) Show that $T$ is linear.

(b) Let $p(X) = 2 + 3X - X^2 + 4X^3$. Determine the image of $p$ under each of the following linear operators: $D$, $T$, $DT$, $TD$, $DT - TD$, $T^2D^2 - D^2T^2$.

(c) Determine those $p \in V$ for which $T(p) = p$.

(d) Determine those $p \in V$ for which $(DT - 2D)(p) = 0$.

(e) Determine those $p \in V$ for which $(DT - TD)^n(p) = D^n(p)$.

9. Consider the following five vectors in the vector space $\mathbb{C}^5$:

$v_1 = (1, 1, 3, -2, 3), \quad v_4 = (0, 3, 1, -3, 1),$
$v_2 = (0, 1, 0, -1, 0), \quad v_5 = (2, -1, -1, -1, -1),$
$v_3 = (2, 3, 6, -5, 6).

(a) Show that the row reduced (echelon) form of the matrix

$$
\begin{pmatrix}
1 & 0 & 2 & 0 & 2 \\
1 & 1 & 3 & 3 & -1 \\
3 & 0 & 6 & 1 & -1 \\
-2 & -1 & -5 & -3 & -1 \\
3 & 0 & 6 & 1 & -1
\end{pmatrix}
$$

is the matrix

$$
\begin{pmatrix}
1 & 0 & 2 & 0 & 2 \\
0 & 1 & 1 & 0 & 18 \\
0 & 0 & 0 & 1 & -7 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
.$$ 

(The columns of the first matrix are simply the vectors $v_1, \ldots, v_5$.)

(b) Conclude that the $v_i$’s are linearly dependent, that the subspace $W$ spanned by them is 3-dimensional, and that $\{v_1, v_2, v_4\}$ is a basis for $W$.

(c) From part (a) conclude that the coefficients $a_1, \ldots, a_5$ of any linear combination

$$
a_1v_1 + a_2v_2 + a_3v_3 + a_4v_5 + a_5v_5 = 0
$$

are the solutions of the linear equations

$$
\begin{align*}
a_1 + 2a_3 + 2a_5 &= 0 \\
a_2 + a_3 + 18a_5 &= 0 \\
a_4 - 7a_5 &= 0.
\end{align*}
$$

Deduce that $a_1, a_2, \text{ and } a_4$ are determined by $a_3$ and $a_5$ by the equations

$$
a_1 = -2a_3 - 2a_5, \quad a_2 = -a_3 - 18a_5, \quad a_4 = 7a_5,
$$

thereby giving all linear relations satisfied by the vectors $v_1, \ldots, v_5$. (There are no further conditions on $a_3$ and $a_5$, so these two can take arbitrary values.)

(d) In particular, show that

$$
2v_1 = -v_2 + v_3 = -18v_2 + 7v_4 + v_5.
$$