1. If \( p \) is the characteristic of a field \( F \), we want to show \( p \) is a prime. Let’s prove by contradiction. If \( p \) is not a prime, then \( p \) is a product of two integers \( m \) and \( n \), i.e., \( p = mn \), and \( 1 < m < p, 1 < n < p \). By the definition of characteristic of a field, we have \( p = 0 \) in the field \( F \). Therefore, \( p = mn = 0 \). We get \( m = 0 \) or \( n = 0 \) since if \( m \neq 0 \), then, by law 7 in the definition of a field, we have \( n = (m^{-1}m)n = m^{-1}0 = 0 \). It is no harm to assume \( m = 0 \). Notice that \( m < p \) and \( m = 0 \). Contradiction to \( p \) is the least number in \( F \) such that \( p = 0 \).

2. Let \( F = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\} \). \( F \) is a field. To prove \( F \) is a field. We need to verify \((F, +, \cdot)\) satisfy the definition of field. That is, verify \((F, +, \cdot)\) satisfies 9 laws of the definition.

It’s easy to verify all laws except law 8. For a nonzero element \( x = a + b\sqrt{2} \in F \), we need to show there exists an inverse \( x^{-1} \) of \( x \), such that \( xx^{-1} = 1 \). Let \( x^{-1} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2} \). We can verify that \( xx^{-1} = 1 \). The only problem here is to mention that our definition of \( x^{-1} \) is well defined if \( x \) is nonzero. In other words, we need to verify \( a^2 - 2b^2 \neq 0 \) if \( a, b \neq 0 \) (\( x \neq 0 \) means \( a \neq 0 \) or \( b \neq 0 \)). This is because \( a \) and \( b \) are rational numbers. We can assume \( a = \frac{m}{n}, b = \frac{t}{s} \), where \( m, n, s, t \in \mathbb{Z} \) and \( \text{g.c.d.}(m, n) = \text{g.c.d.}(s, t) = 1 \). So if \( a^2 - 2b^2 = 0 \), we can get \( m^2t^2 = 2n^2s^2 \). By counting the prime factors of \( m^2t^2 \) and \( 2n^2s^2 \), we found it could not be true when \( m, n, s, t \) are integers.

3. If matrices \( A \) and \( B \) are row equivalent, we want to show \( A \) and \( B \) have the same row-reduced echelon matrix.

By the corollary in page 23, let \( A \) and \( B \) be \( m \times n \) matrices. Then \( B \) is row-equivalent to \( A \) if and only if \( B = PA \) where \( P \) is an invertible \( m \times m \) matrix.

In our problem, since \( A \) and \( B \) are row equivalent, there is an invertible matrix \( P \), such that \( B = PA \). Assume \( R \) is the row-reduced echelon matrix of \( A \), then \( R \) and \( A \) are row equivalent. So there is an invertible matrix \( Q \) such that \( R = QA \). Therefore, \( R = QA = QP^{-1}B \). This means that \( B \) and \( R \) are row equivalent. Since a matrix has a unique row-reduced echelon matrix. So \( R \) is the echelon matrix of \( B \) as well.

If \( A \) is invertible, then \( A^{-1}A = I \). By the corollary above, we get the row-reduced echelon matrix of \( A \) is the identity matrix \( I \).

4. a) Use the same method of Example 16 in textbook page 25. We can get the inverse matrix is

\[
\begin{pmatrix}
13/24 & -5/24 & -1/4 & 1/6 \\
-3/8 & 3/8 & 1/4 & -1/2 \\
-19/24 & 11/24 & -1/4 & -1/6 \\
-1/4 & 1/4 & 1/2 & 0
\end{pmatrix}
\]

b) The solutions are \( x_1 = -\frac{3}{4}c_4 - \frac{39}{4}c_5 + \frac{12}{7}, x_2 = \frac{8}{5}c_4 + \frac{2}{5}c_5 + \frac{22}{7}, x_3 = \frac{8}{5}c_4 - \frac{35}{4}c_5 + \frac{20}{7}, x_4 = c_4, x_5 = c_5 \), where \( c_4, c_5 \) are arbitrary numbers.

5. Let a \( n \times n \) matrix \( A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \). Define the trace of \( A \) by sum of the diagonal entries, i.e., \( \text{Tr}(A) = \sum_{i=1}^{n} a_{ii} \).

We can verify that \( \text{Tr}(AB - BA) = 0 \), but \( \text{Tr}(I) = n \neq 0 \). So there are no such matrices \( A \) and \( B \) over rational number field.

6. a) Let \( A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \);

b) Let \( A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \);

c) Let \( A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \).
7. The second statement comes from the corollary in textbook page 24.

Since $AB = I$, by the second statement, we have $A$ and $B$ are invertible. Since $AB = I$, we times $B$ from left and get $BAB = BI = B$. And we times $B^{-1}$ from right and get $BABB^{-1} = BB^{-1} = I$. And $BABB^{-1} = BA(BB^{-1}) = BA$. Hence we get $BA = I$.

8. The answer is $(7^2 - 1)(7^2 - 7) = 48 \times 42 = 2016$. First row of the matrix couldn’t be 0, so there are $7^2 - 1$ choices. Second row couldn’t be linear dependent with the first row, so there are $(7^2 - 7)$ choices. Therefore, in total, there are $(7^2 - 1)(7^2 - 7) = 48 \times 42 = 2016$ $2 \times 2$ matrices over the finite field of 7 elements.