1. a) Suppose \( \text{dim} V = n, \{ v_1, \ldots, v_m \}, m > n \) is a collection of vectors in \( V \). Suppose the coordinates of \( v_i = (a_{i1}, \ldots, a_{in}), i = 1, \ldots, m \), then \( k_1v_1 + \ldots + k_nv_m = 0 \) corresponds a linear system

\[
\begin{pmatrix}
a_{11} & a_{21} & \cdots & a_{m1} \\
a_{12} & a_{22} & \cdots & a_{m2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_2 \\
\vdots \\
k_m
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

Since the number of equations, which is \( n \), is less than the number of unknowns, which is \( m \), we have the homogeneous system must have non-trivial solutions. That is, there exists a non-zero solution \( k_1, \ldots, k_m \). So \( \{ v_1, \ldots, v_m \} \) are linearly dependent.

b) \( T \) is injective means \( T \) is one-to-one when \( T \) is viewed as a map. Since \( T \) is linear, we get \( T(0) = 0 \), that is, \( 0 \in \ker T \). Since \( T \) is injective, we have \( \ker T = \{ 0 \} \).

On the other hand, if \( \ker T = \{ 0 \} \), and \( T(v_1) = T(v_2) \), then \( T(v_1 - v_2) = T(v_1) - T(v_2) = 0 \) since \( T \) is linear. So \( v_1 - v_2 \in \ker T \). By assumption, \( \ker T = \{ 0 \} \). So \( v_1 - v_2 = 0 \), i.e., \( v_1 = v_2 \). So \( T \) is injective.

Let \( n = \text{dim} V = \text{dim} W \) and \( \{ e_1, \ldots, e_n \} \) be a basis of \( V \).

If \( T \) is injective. Claim \( \{ T(e_1), \ldots, T(e_n) \} \) is a basis of \( W \). Since \( \text{dim} W = n \), we only need to show \( \{ T(e_1), \ldots, T(e_n) \} \) is linearly independent. Indeed, if \( k_1T(e_1) + k_2T(e_2) + \ldots + k_nT(e_n) = 0 \), then \( T(k_1e_1 + k_2e_2 + \ldots + k_ne_n) = 0 \). So \( k_1e_1 + k_2e_2 + \ldots + k_ne_n \in \ker T \). Since \( \ker T = \{ 0 \} \), we have \( k_1e_1 + k_2e_2 + \ldots + k_ne_n = 0 \). And \( \{ e_1, \ldots, e_n \} \) are linearly independent, so \( k_1 = k_2 = \ldots = k_n = 0 \). That is \( \{ T(e_1), \ldots, T(e_n) \} \) are linearly independent. Hence \( \text{dim} \text{Im} T \geq n \). And since \( \text{Im} T \) is a subspace of \( W \) and they have same dimension. \( \text{dim} T = W \), i.e., \( T \) is surjective.

On the other hand. If \( T \) is surjective, i.e., \( \text{Im} T = W \). If \( \ker T \neq \{ 0 \} \), then there exists a nonzero vector \( v \in \ker T \). So we can extend it to a basis of \( V \), say \( \{ e_1 = v, e_2, \ldots, e_n \} \). Since \( \text{Im} T \) is spanned by \( T(e_2), \ldots, T(e_n) \), we get \( \text{dim} \text{Im} T \) at most \( n - 1 \). So \( T \) cannot be surjective. Contradiction! So \( \ker T = \{ 0 \} \), i.e., \( T \) is injective.

If \( T \) is a linear transformation, and, as a map, \( T \) is one-to-one and onto, we want to show \( T^{-1} \) is a linear transformation as well. For every vector \( w \in W \), since \( T \) is surjective and injective, there exists a unique vector \( v \in V \), such that \( T(v) = w \). Define \( T^{-1}(w) = v \). Under this definition, we want to show \( T^{-1} \) is linear, i.e., \( T^{-1}(cv_1 + w_2) = cT^{-1}(w_1) + T^{-1}(w_2) \) for any \( w_1, w_2 \in W, c \in F \). Indeed, assume \( T(v_1) = w_1 \) and \( T(v_2) = w_2 \). So \( T(cv_1 + v_2) = cw_1 + w_2 \), i.e., \( T^{-1}(cw_1 + w_2) = cv_1 + v_2 \). Moreover, \( cT^{-1}(w_1) + T^{-1}(w_2) = cv_1 + v_2 \). So \( T^{-1} \) is linear.

c) It is easy to check \( \text{Im} T \) is a subspace of \( W \) by the definition of subspace. \( V/\ker T \) consists of elements \( v + \ker T, \forall v \in V \). Let \( \varphi : V/\ker T \rightarrow W \) by \( \varphi(v + \ker T) = T(v) \). \( \varphi \) is well defined since \( \varphi(v + \ker T) = T(v) = T(v') = \varphi(v' + \ker T) \) if \( v, v' \) are in the same class \( v + \ker T \). Indeed, if \( v, v' \) are in the same class \( v + \ker T \), then \( v - v' \in \ker T \). So \( T(v) = T(v') \). Second, we need to show \( \varphi \) is linear. This is easy from the definition. Third, we need to show \( \varphi \) is one-to-one and onto. Onto is clear. If \( \varphi(v + \ker T) = T(v) = 0 \). Then \( v \in \ker T \). So \( v + \ker T \) is the zero element in the quotient vector space \( V/\ker T \).

d) Let \( S \) consists of all elements \( v \), such that \( T(v) = w \). Since \( T(v_0) = w \), we can easily see that \( v_0 + \ker T \subset S \). On the other hand, if \( v' \in S \), then \( T(v') = T(v) \). So \( T(v - v') = 0 \), that is \( v - v' \in \ker T \). Therefore, \( v' \in v + \ker T \).

e) Let \( \{ e_1, \ldots, e_k \} \) is a basis of \( ker T \), then it can be extended to a basis of \( V \), say \( \{ e_1, \ldots, e_k, e_{k+1}, \ldots, e_n \} \), where \( n = \text{dim} V \). Then \( \{ T(e_{k+1}), \ldots, T(e_n) \} \) spans \( \text{Im} T \). And actually \( \{ T(e_{k+1}), \ldots, T(e_n) \} \) is linearly independent. Otherwise, there exists nonzero \( a_{k+1}, \ldots, a_n \) such that \( a_{k+1}T(e_{k+1}) + \ldots + a_nT(e_n) = 0 \). So \( a_{k+1}e_{k+1} + \ldots + a_ne_n \in \ker T \). Since \( \{ e_1, \ldots, e_k \} \) is a basis of \( ker T \), we have \( a_{k+1}e_{k+1} + \ldots + a_ne_n = a_1e_1 + \ldots + a_ke_k \). And we get \( a_1 = a_2 = \ldots = a_k = a_{k+1} = \ldots = a_n = 0 \) since \( \{ e_1, \ldots, e_k, e_{k+1}, \ldots, e_n \} \) is linearly independent. Contradiction! So \( \text{dim} ker T = k \) and \( \text{dim} \text{im} T = n - k \), Therefore, \( \text{dim}(\ker T) + \text{dim}(\text{Im} T) = k + (n - k) = n = \text{dim} V \).

For any subspace \( U \) of \( V \), we can construct a linear transformation \( T \), such that \( \ker T = U \). For instance, let \( \{ e_1, \ldots, e_k \} \) be a basis of \( U \), and extend it to be a basis \( \{ e_1, \ldots, e_k, e_{k+1}, \ldots, e_n \} \) of \( V \). Let
$T(e_1) = \ldots = T(e_k) = 0$ and $T(e_{k+1}) = \ldots = T(e_n) = 1$. Then $\ker T = U$. So we have $\dim V = \dim \ker T + \dim (\text{Im} T) = \dim U + \dim (V/\ker T) = \dim U + \dim (V/U)$.

If $\dim V > \dim W$, then $\dim \ker T = \dim V - \dim W > 0$. So $\ker T \neq 0$.

2. ker $L = \{(a_1,0,0,\ldots)\}, a_1 \in \mathbb{R}$. Im $L = V$ since every sequence $(b_1,b_2,\ldots)$ is the image of $(1,b_1,b_2,\ldots)$.

ker $R = 0$, but Im $R \neq V$ since the first entry of the sequence in Im $R$ is 0.

3. a) Couldn’t. Since $\dim U > \dim V$, we have ker $S \neq 0$, i.e., $S$ is not injective. However, if $TS$ is invertible, $S$ has to be injective.

b) Follows from the definition of matrix of a linear transformation with respect to the base directly. (Theorem 13 in the textbook page 90)

4. a) Let

$$A = \begin{pmatrix} 0 & 3 & 1 \\ 2 & 0 & -1 \\ -1 & -2 & 0 \end{pmatrix}, P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$  

Then the matrix of $T$ with respect to the basis $(0,1,-1),(1,-1,1),(-1,1,0)$ is

$$P^{-1}AP = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 2 & -3 \end{pmatrix}.$$

b) Let

$$A = \begin{pmatrix} 6 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}.$$  

If $A$ and $B$ are similar, then there exists an invertible matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, such that $B = P^{-1}AP$, i.e., $AP = PB$. So we get

$$AP = \begin{pmatrix} 6 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 6a & 6b \\ -c & -d \end{pmatrix} = PB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} a + 5b & 2a + 4b \\ c + 5d & 2c + 4d \end{pmatrix}$$

So we get a linear system about $a,b,c,d$, by comparing the corresponding entries,

$$\begin{cases} 6a = a + 5b \\ 6b = 2a + 4b \\ -c = c + 5d \\ -d = 2c + 4d \end{cases}$$

There are infinitely many solutions. We can take $P = \begin{pmatrix} 1 & 5 \\ 1 & -2 \end{pmatrix}$.

5. a) By computation, we get $T(1) = 12, T(X) = 6X, T(X^2) = 2X^2, T(X^3) = 0, T(X^4) = 0, T(X^5) = 2X^5$. So the matrix $M_4$ of $D$ respect to the basis $\{1,X,X^2,X^3,X^4,X^5\}$ is

$$M_4 = \begin{pmatrix} 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By computation, we get

$T(1) = 12,$

$T(1 + X) = 6X + 12 = 6 + 6(1 + X),$ 

$T(1 + X + X^2) = 2X^2 + 6X + 12 = 6 + 4(1 + X) + 2(1 + X + X^2),$

$T(1 + X + X^2 + X^3) = T(1) + T(X) + T(X^2) + T(X^3) = 2X^2 + 6X + 12 = 6 + 4(1 + X) + 2(1 + X + X^2),$

$T(1 + X + X^2 + X^3 + X^4) = 2X^2 + 6X + 12 = 6 + 4(1 + X) + 2(1 + X + X^2),$

$T(1 + X + X^2 + X^3 + X^4 + X^5) = 2X^5 + 2X^2 + 6X + 12$

$= 6 + 4(1 + X) + 2(1 + X + X^2) - 2(1 + X + X^2 + X^3 + X^4) + 2(1 + X + X^2 + X^3 + X^4 + X^5).$
So the matrix $M_2$ of $D$ respect to the basis $\{1, 1 + X, 1 + X + X^2, 1 + X + X^2 + X^3, 1 + X + X^2 + X^3 + X^4, 1 + X + X^2 + X^3 + X^4 + X^5\}$ is

$$
M_2 = \begin{pmatrix}
12 & 6 & 6 & 6 & 6 \\
0 & 6 & 4 & 4 & 4 \\
0 & 0 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}
$$

b) From a), we can see that $\{X^3, X^4\}$ is a basis of $\text{ker} \, D$ and $\{1, X, X^2, X^5\}$ is a basis of $\text{Im} \, D$.

6) Since $\{a_n\}$ is a convergent sequence, we can see that $b_n = a_n - \lim_{m \to \infty} a_m$ is a convergent sequence as well and its limit is 0. So $\text{Im} \, T$ consists of all limit 0 sequences in $V$, i.e., $\text{Im} \, T = \{(a_1, a_2, \ldots) | \lim_{m \to \infty} a_n = 0\}$.

For any two convergent sequences $\{a_n\}, \{a_n'\}, c \in F$,

$$
T(c\{a_n\} + \{a_n'\}) = T(c\{a_n + a_n'\}) = \{ca_n + a_n' - \lim_{m \to \infty} (ca_m + a'_m)\} = c\{a_n - \lim_{m \to \infty} a_m\} + \{a_n' - \lim_{m \to \infty} a'_m\} = cT(\{a_n\}) + T(\{a_n'\})
$$

So $T$ is linear.

Note that $(0, 0, 0, \ldots)$ is the zero element in $V$. So $\text{ker} \, T = \{(a, a, a, \ldots) | a \in F\}$.

7. a) Let $\{e_1, \ldots, e_k\}$ be a basis of $\text{Im} \, T$. And $\{e_1^*, \ldots, e_k^*\}$ be its dual, i.e., $e^*_i(e_j) = \delta_{ij}, i, j = 1, \ldots, k$. Then $T^*(e^*_1), \ldots, T^*(e^*_k) \in \text{Im} \, T^* \subset V^*$. Now we are trying to show $\{T^*(e^*_1), \ldots, T^*(e^*_k)\}$ is linear independent in $\text{Im} \, T^*$. If $a_1 T^*(e^*_1) + \ldots + a_k T^*(e^*_k) = 0$, then for any $v \in V$,

$$
(a_1 T^*(e^*_1)(v) + \ldots + a_k T^*(e^*_k)(v)) = a_1 T^*(e^*_1)(v) + \ldots + a_k T^*(e^*_k)(v) = a_1 e_1^*(T(v)) + \ldots + a_k e_k^*(T(v)) = 0.
$$

Since the equality is true for any $v \in V$, we can choose $v_1, \ldots, v_k$, such that $e_i = T(v_i), i = 1, \ldots, k$ (notice that $\{e_i\}$ is in the image of $T$). Plug in $v_i, i = 1, \ldots, k$ in the equality, we get $a_1 = \ldots = a_k = 0$. Hence $\{T^*(e^*_1), \ldots, T^*(e^*_k)\}$ is linear independent in $\text{Im} \, T^*$. So we have $\dim(\text{Im} \, T^*) \geq \dim(\text{Im} \, T)$. Similarly, we have $\dim(\text{Im} \, T^{**}) \geq \dim(\text{Im} \, T^*)$. Indeed, let $S = T^*$, and use the same argument above.

By theorem 23 in the textbook page 113, if $A$ is the matrix of $T$ relative to $\mathcal{B}, \mathcal{B}'$, where $\mathcal{B}$ is a basis of $V$ and $\mathcal{B}'$ is a basis of $W$, then the matrix of $T^*$ relative to the dual basis $\mathcal{B}^{**}, \mathcal{B}'^*$ of $\mathcal{B}, \mathcal{B}'$ is the transpose $A^t$ of $A$.

So $(A^t)^t = A$ is the matrix of $T^{**}$ relative to the basis $\mathcal{B}^{**}, \mathcal{B}'^{**}$. Therefore,

$$
\dim(\text{Im} \, T^{**}) = \text{column rank}(A) = \dim(\text{Im} \, T).
$$

So we have $\dim(\text{Im} \, T^{**}) \geq \dim(\text{Im} \, T^*) \geq \dim(\text{Im} \, T)$ and $\dim(\text{Im} \, T^{**}) = \dim(\text{Im} \, T)$. Hence $\dim(\text{Im} \, T^*) = \dim(\text{Im} \, T)$.

b) From a), we have $\dim(\text{Im} \, T^*) = \dim(\text{Im} \, T)$. And apply theorem 23 in the textbook page 113, we get

$$
\text{column rank}(A) = \text{rank}(T) = \dim(\text{Im} \, T) = \dim(\text{Im} \, T^*) = \text{rank} T^* = \text{column rank}(A^t) = \text{row rank}(A).
$$

8. Assume $\text{Im} \, T$ and $\text{ker} \, T$ are both finite dimensional. Let $\{v_1, \ldots, v_k\}$ be a basis of $V$. And it can be extended to a basis $\mathcal{B}$ of $V$. Since $V$ is infinite dimensional, we have $\mathcal{B} \setminus \{v_1, \ldots, v_k\}$ is infinite. And we can see that $\text{Im} \, T$ is spanned by the images of $\mathcal{B} \setminus \{v_1, \ldots, v_k\}$ (see of proof of problem 1). By our assumption, $\text{Im} \, T$ is finite dimensional, so there exist $v_{k+1}, \ldots, v_m \in \mathcal{B} \setminus \{v_1, \ldots, v_k\}$ such that $T(v_{k+1}), \ldots, T(v_m)$ are linear dependent. So there exist nonzero $a_{k+1}, \ldots, a_m$, such that $a_{k+1}T(v_{k+1}) + \ldots + a_mT(v_m) = 0$. So we get $a_{k+1}v_{k+1} + \ldots + a_mv_m \in \text{ker} \, T$. So it can be represented by the basis of $\text{ker} \, T$, i.e., $a_{k+1}v_{k+1} + \ldots + a_mv_m = a_1v_1 + \ldots + a_kv_k$. However, since $v_1, \ldots, v_k, v_{k+1}, \ldots, v_m$ is a part of basis, they are linearly independent. So $a_1 = a_2 = \ldots = a_k = a_{k+1} = \ldots = a_m = 0$. Contradicts to $a_{k+1}, \ldots, a_m$ are not all zero.