Section 1.2.3. Exercises.

1. To prove such proposition “· · · is either A or B”, the standard way is to prove that if it is not A, then it is B.

   Let S be a subset of \( \mathbb{N} \), where \( \mathbb{N} \) denotes the natural numbers. If S is not a finite set, we want to show S is countable.

   **Proof I.** (using the ordering of \( \mathbb{N} \))

   **Least-integer principle:** Every non-empty set of natural numbers has a least number.

   Since S is not a finite set, S must be an infinite set, and thus S is nonempty in \( \mathbb{N} \). By Least-integer principle, suppose \( a_1 \in S \) is the least element. Now we consider \( S \setminus \{ a_1 \} \). Since S is infinite, \( S \setminus \{ a_1 \} \) is an infinite subset in \( \mathbb{N} \). Again by Least-integer principle, there is a least element \( a_2 \in S \setminus \{ a_1 \} \). Keep going, we can enumerate S by ordering, i.e.,

   \[ S = \{ a_1, a_2, \ldots \} , \]

   where \( a_i \) is the least element in \( S \setminus \{ a_1, a_2, \ldots, a_{i-1} \} \).

   **Proof II.** (using Cantor-Berstein-Schroeder’s Th)

   Suppose S is an infinite subset of \( \mathbb{N} \). Then we have a natural inclusion \( f : S \rightarrow \mathbb{N} \), which is one-to-one. By the fact that any infinite set contains a countable subset (the proof is in course notes 2), S has a countable subset \( E \), since S is infinite. So we have a one-to-one map \( g : E \sim E \rightarrow S \). By Cantor-Berstein-Schroeder’s Th, \( S \sim \mathbb{N} \).

2. The set of all finite subsets of \( \mathbb{N} \) is countable.

   **Proof.** Let \( A_i \) be the set of subsets of \( \mathbb{N} \) consisting of \( i \) elements, \( A \) the set of all finite subsets of \( \mathbb{N} \). Then \( A \) is a disjoint union of \( A_i \), i.e.,

   \[ A = \bigcup_{i=0}^{\infty} A_i. \]

   Since there is a natural onto map \( f : \mathbb{N} \times \cdots \times \mathbb{N} \) (cross products \( i \) times) \( \rightarrow \mathbb{N}((a_1, a_2, \ldots, a_i) \mapsto \{ a_1, a_2, \ldots, a_i \} \), we get \( A_i \) is countable for \( i > 0 \) by the simple Lemma in page 9 (Simple Lemma: if there is a mapping of the natural numbers onto a set \( U \), then \( U \) is either finite or countable. You can use Exercise 1 to prove the lemma. Try to make it.). So \( A = \bigcup_{i=0}^{\infty} A_i \) is countable.

3. Let \( \mathbb{Q}_+ \) be the set of all positive rational numbers, \( \mathbb{Q}_- \) the set of all negative rational numbers. So \( \mathbb{Q} = \mathbb{Q}_+ \cup \mathbb{Q}_- \cup \{0\} \). Define a map \( f : \mathbb{N} \times \cdots \times \mathbb{N} \rightarrow \mathbb{Q}_+ \) by \( f(p, q) = p/q \). Then we can verify that \( f \) is onto. Indeed, every positive rational number has form \( p/q \) for some natural numbers \( p \) and \( q \). Since \( \mathbb{N} \) is countable, \( \mathbb{N} \times \cdots \times \mathbb{N} \) is countable as well (see the example in page 10 textbook). Since \( f \) is onto, we get \( \mathbb{Q}_+ \) is countable by the simple lemma in page 9.

   Since \( \mathbb{Q}_- \sim \mathbb{Q}_+ \) (check the map \( g : \mathbb{Q}_+ \rightarrow \mathbb{Q}_- \) by \( g(r) = -r, r \in \mathbb{Q}_+ \) is one-to-one and onto), we get \( \mathbb{Q}_- \) is countable as well. Hence \( \mathbb{Q} = \mathbb{Q}_+ \cup \mathbb{Q}_- \cup \{0\} \) is countable (countable union a finite set is countable. See the next problem for a proof).

4. Let \( A \) be an uncountable set, \( U \) a countable subset of \( A \). Let \( V = A \setminus U \). Then \( V \) is uncountable. Indeed, we can prove by contradiction.

   If \( V \) is not uncountable, then \( V \) is either finite or countable. If \( V \) is finite, \( V = \{ a_1, \ldots, a_n \} \). Since \( U \) is countable, assume \( U = \{ b_1, b_2, b_3, \ldots \} \). Then \( A = U \cup V = \{ a_1, \ldots, a_n, b_1, b_2, b_3, \ldots \} \) is countable. It contradicts to the assumption that \( A \) is uncountable. If \( V \) is countable, then \( A = U \cup V \) is countable. Indeed, there is a one-to-one and onto map \( f : U \rightarrow \mathbb{N} \). Similarly, there
is a one-to-one map \( g : V \rightarrow \mathbb{Z}_{\leq 0} \), where \( \mathbb{Z}_{\leq 0} \) denotes all non-positive integers (notice that \( \mathbb{Z}_{\leq 0} \) is countable). So \( \alpha : U \cup V \rightarrow \mathbb{Z} \) by

\[
\alpha(x) = \begin{cases} 
  f(x) & x \in U \\
  g(x) & x \in V
\end{cases}
\]

We can easily verify that \( \alpha \) is one-to-one and onto (notice that \( U \cap V = \emptyset \), so \( \alpha \) is well defined). Hence \( U \cap V \) is countable. Contradiction!

5. We want to show that \( A_1 \times A_2 \times \cdots \) is uncountable if each \( A_i \) is countable, or more generally if each \( A_i \) has at least two elements.

We are going to show the second proposition. Since each \( A_i \) has at least two elements, we can suppose \( a_i, b_i \) are two distinct elements in \( A_i \). Construct a map \( f : 2^\mathbb{N} \rightarrow A_1 \times A_2 \times A_3 \times \cdots \) by \( f(S) = (x_1, x_2, \ldots) \), where \( x_i \) is defined by

\[
x_i = \begin{cases} 
  a_i, & i \in S \\
  b_i, & i \notin S
\end{cases}
\]

and \( S \) is an arbitrary subset of \( \mathbb{N} \). It is easy to see that \( f \) is one-to-one. So \( A_1 \times A_2 \times \cdots \) is uncountable. Otherwise \( A_1 \times A_2 \times \cdots \) is finite or countable, and so is the subset \( 2^\mathbb{N} \) by Exercise 1. Contradiction!

6. Let \( A_k \) be the subset of \( A \) given by the solutions to \( f(a) = k \) (people always use \( f^{-1}(k) \) to denote the set). Then \( A \) is the disjoint union of \( A_k \), i.e. \( A = \bigcup_{k=1}^{\infty} A_k \). If \( A \) is not finite, then \( A_k \neq \emptyset \) for infinity many \( k \) because \( A_k \) is finite by assumption. So \( A \) is a subset of \( \bigcup_{k=1}^{\infty} N_k \), where each \( N_k \sim \mathbb{N} \) but none of them intersect. We know that \( \bigcup_{k=1}^{\infty} N_k \) is countable. By Exercise 1, we get \( A \) is countable as well.

7. See the course notes for the proof. (diagonalization argument)

**Extra Problem.** Show that \( \mathbb{N}^\mathbb{N} = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots \sim \mathbb{R} \).

**Proof.** Since \( \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots \sim \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \cdots \), we get \( \mathbb{R} \) is a subset of \( \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \cdots \) and therefore \( \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots \) by the definition of real numbers. On the other hand, we can associate every sequence of natural numbers to a real number. Let \( (a_1, a_2, a_3, \ldots) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots \) is a sequence of natural number. Then it can be associate to a real number (decimal expression) \( 0.0 \cdots 0/0/0 \cdots 01 \cdots \) (first has \( a_1 \) 0s, then 1 then \( a_2 \) 0s then 1). For example, sequence \( (1, 2, 3, 0, 1, 0, \ldots) \) corresponds to \( 0.01001000101\ldots \). The map we defined is one-to-one. So we get \( \mathbb{N}^\mathbb{N} = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots \sim \mathbb{R} \).