Math 406 - Analysis II
Final Exam — May, 2012

• Let $S$ be a totally bounded subset in a complete metric space $(M,d)$. Let $f : M \rightarrow S$ be a map satisfying $d(f(x), f(y)) < d(x, y)$ for any $x, y \in M$. Prove that $f$ has a unique fixed point.

• Let $(M,d)$ be a metric space. Show that a sequence $\{s_n\} \subset M$ is Cauchy if and only if
$$\liminf_{n \to \infty} \left( \sup_{j>0} d(s_n, s_{n+j}) \right) = 0.$$

• Prove that the metric space $M$ is compact if and only if any continuous function on $M$ is bounded.

• (generalized Riemann-Lebesgue lemma) Let $g_n(x)$ be a sequence of measurable functions on $[a, b], -\infty < a < b < \infty$. Suppose there exists $M > 0$ such that $|g_n(x)| \leq M$ and for any $c \in [a, b]$, one has
$$\lim_{n \to \infty} \int_a^c g_n(x) \, dx = 0.$$

Then prove
$$\lim_{n \to \infty} \int_a^b f(x)g_n(x) \, dx = 0$$
for any $f \in L^1([a,b])$.

• (Holder inequality) Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p, g \in L^q$. Then $fg \in L^1$ and moreover, one has
$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

• (Minkowski inequality) Let $1 \leq p \leq \infty$ and $f, g \in L^p$. Then
$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

• Read Chapter 13. State and Prove the implicit function theorem and inverse function theorem in $\mathbb{R}^m$ for $m > 1$. 